Abstract Interpretation

SOS, Master Recherche Science Informatique, U. Rennes

Thomas Jensen

(slides by David Pichardie, Thomas Jensen)

Static program analysis

The goals of static program analysis

- to prove properties about the run-time behaviour of a program
- in a fully automatic way
- without actually executing this program

Abstract interpretation : a theory of **semantic approximation**, which unifies a large variety of static analyses.

Abstract interpretation

- formalises the approximated analysis of programs,
- allows to compare the relative precision of analyses,
- facilitates the design of sophisticated analyses.

A flavor of abstract interpretation

Abstract interpretation executes programs on **state properties** instead of states.

Collecting semantics

- A state property is a subset in $\mathcal{P}(\mathbb{Z}^2)$ of $(\mathtt{x}, \mathtt{y})$ values.
- When a point is reached for a second time we make a union with the previous property.
- We "execute" the program until stability
  - It may take an infinite number of steps...
  - But the limit always exists (explained later)

```
x = 0; y = 0;
```
$\{(0,0),(1,0),(1,2),\ldots\}$
```
while (x<6) {
  if (?) {
```
$\{(0,0),(1,0),(1,2),\ldots\}$
```
    y = y+2;
```
$\{(0,2),(1,2),(1,4),\ldots\}$
```
  };
```
$\{(0,0),(0,2),(1,0),(1,2),(1,4),\ldots\}$
```
  x = x+1;
```
$\{(1,0),(1,2),(2,0),(2,2),(2,4),\ldots\}$
```
}
```
$\{(6,0),(6,2),(6,4),(6,6),\ldots\}$

A flavor of abstract interpretation

Abstract interpretation executes programs on **state properties** instead of states.

Approximation

- The set of manipulated properties may be restricted to ensure computability of the semantics.
  Example : sign of variables

$$\begin{array}{lll} P & ::= & x\ C\ 0 \wedge y\ C\ 0 \\ C & ::= & < \mid \leqslant \mid = \mid > \mid \geqslant \end{array}$$

- To stay in the domain of selected properties, we over-approximate the concrete properties.

```
x = 0; y = 0;
```
$x \geqslant 0 \wedge y \geqslant 0$
```
while (x<6) {
  if (?) {
```
$x \geqslant 0 \wedge y \geqslant 0$
```
    y = y+2;
```
$x \geqslant 0 \wedge y > 0$
```
  };
```
$x \geqslant 0 \wedge y \geqslant 0$
```
  x = x+1;
```
$x > 0 \wedge y \geqslant 0$
```
}
```
$x > 0 \wedge y \geqslant 0$

Another example : the interval analysis

For each point k and integer variable x, we infer an interval to which x must belong.

Example : insertion sort, array access verification (buffer overflow)

```
assume(T.length=100); i=1;
                                        {i ∈ [1,100]}
while (i<T.length) {
                                        {i ∈ [1,99]}
    p = T[i]; j = i-1;
                                        {i ∈ [1,99], j ∈ [−1,98]}
    while (0<=j and T[j]>p) {
                                        {i ∈ [1,99], j ∈ [0,98]}
        T[j+1]=T[j]; j = j-1;
                                        {i ∈ [1,99], j ∈ [−1,97]}
    };
                                        {i ∈ [1,99], j ∈ [−1,98]}
    T[j+1]=p; i = i+1;
                                        {i ∈ [2,100], j = [−1,98]}
};
                                        {i = 100}
```

Another example : the polyhedral analysis

For each point k, infer linear equality and inequality relationships among variables.

Example : insertion sort, array access verification (buffer overflow)

```
assume(T.length>=1); i=1;
                                        {1 ⩽ i ⩽ T.length}
while i<T.length {
                                        {1 ⩽ i ⩽ T.length − 1}
    p = T[i]; j = i-1;
                                        {1 ⩽ i ⩽ T.length − 1 ∧ −1 ⩽ j ⩽ i − 1}
    while 0<=j and T[j]>p {
                                        {1 ⩽ i ⩽ T.length − 1 ∧ 0 ⩽ j ⩽ i − 1}
        T[j+1]=T[j]; j = j-1;
                                        {1 ⩽ i ⩽ T.length − 1 ∧ −1 ⩽ j ⩽ i − 2}
    };
                                        {1 ⩽ i ⩽ T.length − 1 ∧ −1 ⩽ j ⩽ i − 1}
    T[j+1]=p; i = i+1;
                                        {2 ⩽ i ⩽ T.length + 1 ∧ −1 ⩽ j ⩽ i − 2}
};
                                        {i = T.length}
```

Outline

1. Introduction
2. Control flow graph
3. Collecting semantics
4. Approximate analysis : an informal presentation
5. Abstraction by intervals
6. Widening and narrowing
7. Galois connections
8. Polyhedral abstract interpretation
9. References

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While syntax

$$
\begin{array}{rll}
Exp ::= & n & n \in \mathbb{Z} \\
\mid & ? & \\
\mid & x & x \in \mathbb{V} \\
\mid & Exp\ o\ Exp & o \in \{+, -, \times\} \\
 & & \\
test ::= & Exp\ c\ Exp & c \in \{=, \neq, <, \leqslant\} \\
\mid & test\ \textbf{and}\ test & \\
\mid & test\ \textbf{or}\ test & \\
 & & \\
Stm ::= & {}^{l}[x := Exp] & l \in \mathbb{P} \\
\mid & {}^{l}[\texttt{skip}] & \\
\mid & \texttt{if}\ {}^{l}[test]\ \{\ Stm\ \}\ \{\ Stm\ \} & \\
\mid & \texttt{while}\ {}^{l}[test]\ \{\ Stm\ \} & \\
\mid & Stm\ ;\ Stm & \\
Prog ::= & [Stm]^{end} & end \in \mathbb{P}
\end{array}
$$

$\mathbb{P}$: set of program points $\quad$ $\mathbb{V}$: set of program variables

While syntax : example

$$
\begin{array}{l}
[^0[x :=?];\\
\texttt{if}\ ^1[x < 0]\ \{\\
\qquad \texttt{while}\ ^2[x < 0]\ \{\\
\qquad\qquad ^3[x := x + 1];\\
\qquad \};\\
\qquad ^4[y := x];\\
\}\ \texttt{else}\ \{\\
\qquad ^5[y := 0];\\
\};]^6
\end{array}
$$

While semantics

Semantic domains

$$\begin{array}{rcl} \mathit{Env} & \stackrel{\text{def}}{=} & \mathbb{V} \to \mathbb{Z} \\ \mathit{State} & \stackrel{\text{def}}{=} & \mathbb{P} \times \mathit{Env} \end{array}$$

Semantics of expressions

$$\mathcal{A}[\![e]\!]\,\rho \in \mathcal{P}(\mathbb{Z}), \quad e \in \mathit{Exp},\ \rho \in \mathit{Env}$$

$$\begin{array}{lcl} \mathcal{A}[\![n]\!]\,\rho & = & \{\, n \,\} \\ \mathcal{A}[\![?]\!]\,\rho & = & \mathbb{Z} \\ \mathcal{A}[\![x]\!]\,\rho & = & \{\, \rho(x) \,\},\ x \in \mathbb{V} \\ \mathcal{A}[\![e_1\ o\ e_2]\!]\,\rho & = & \{\, v_1\, o\, v_2 \mid v_1 \in \mathcal{A}[\![e_1]\!]\,\rho,\ v_2 \in \mathcal{A}[\![e_2]\!]\,\rho \,\} \\ & & o \in \{+, -, \times\} \end{array}$$

Remark : $\mathcal{A}[\![\cdot]\!]\,\rho$ is non-deterministic because of the expression ?.

Semantics of tests

$$\mathcal{B}[\![t]\!]\rho \in \mathcal{P}(\mathbb{B}), \quad t \in \mathit{test},\ \rho \in \mathit{Env} \quad \mathbb{B} = \{\mathbf{tt}, \mathbf{ff}\}$$

$$\frac{v_1 \in \mathcal{A}[\![e_1]\!]\rho \quad v_2 \in \mathcal{A}[\![e_2]\!]\rho \quad v_1\ \bar{c}\ v_2}{\mathbf{tt} \in \mathcal{B}[\![e_1\ c\ e_2]\!]\rho}$$

$$\frac{v_1 \in \mathcal{A}[\![e_1]\!]\rho \quad v_2 \in \mathcal{A}[\![e_2]\!]\rho \quad \neg(v_1\ \bar{c}\ v_2)}{\mathbf{ff} \in \mathcal{B}[\![e_1\ c\ e_2]\!]\rho}$$

$$\frac{b_1 \in \mathcal{B}[\![t_1]\!]\rho \quad b_2 \in \mathcal{B}[\![t_2]\!]\rho}{b_1 \wedge_{\mathbb{B}} b_2 \in \mathcal{B}[\![t_1\ \mathbf{and}\ t_2]\!]\rho}$$

$$\frac{b_1 \in \mathcal{B}[\![t_1]\!]\rho \quad b_2 \in \mathcal{B}[\![t_2]\!]\rho}{b_1 \vee_{\mathbb{B}} b_2 \in \mathcal{B}[\![t_1\ \mathbf{or}\ t_2]\!]\rho}$$

Structural Operational Semantics

Small-step semantics

$$\frac{v \in \mathcal{A}[\![a]\!]\rho}{({}^{l}[x := a], \rho) \Rightarrow \rho[x \mapsto v]} \qquad \frac{}{({}^{l}[\mathtt{skip}], \rho) \Rightarrow \rho}$$

$$\frac{(S_1, \rho) \Rightarrow \rho'}{(S_1 \,;\, S_2, \rho) \Rightarrow (S_2, \rho')} \qquad \frac{(S_1, \rho) \Rightarrow (S_1', \rho')}{(S_1 \,;\, S_2, \rho) \Rightarrow (S_1' \,;\, S_2, \rho')}$$

$$\frac{\mathbf{tt} \in \mathcal{B}[\![b]\!]\rho}{(\mathtt{if}\ {}^{l}[b]\ \mathtt{then}\ S_1\ \mathtt{else}\ S_2, \rho) \Rightarrow (S_1, \rho)}$$

$$\frac{\mathbf{ff} \in \mathcal{B}[\![b]\!]\rho}{(\mathtt{if}\ {}^{l}[b]\ \mathtt{then}\ S_1\ \mathtt{else}\ S_2, \rho) \Rightarrow (S_2, \rho)}$$

$$\frac{\mathbf{tt} \in \mathcal{B}[\![b]\!]\rho}{(\mathtt{while}\ {}^{l}[b]\ \mathtt{do}\ S, \rho) \Rightarrow (S \,;\, \mathtt{while}\ {}^{l}[b]\ \mathtt{do}\ S, \rho)}$$

$$\frac{\mathbf{ff} \in \mathcal{B}[\![b]\!]\rho}{(\mathtt{while}\ {}^{l}[b]\ \mathtt{do}\ S, \rho) \Rightarrow \rho}$$

Reachable states

Reachable states : the set of pairs of program points and states (k, ρ) such that execution of program P reaches program point k with state ρ.

Formally :

$$[\![[P]^{\text{end}}]\!]_{\text{SOS}} = \left\{ (k, \rho) \;\middle|\; \begin{array}{l} \exists \rho_0 \in Env, \\ \quad \exists S \in Stm,\ (P, \rho_0) \Rightarrow^* (S, \rho) \text{ and } k = \text{entry}(S) \\ \quad \text{or } (P, \rho_0) \Rightarrow^* \rho \text{ and } k = \text{end} \end{array} \right\}$$

A control flow graph representation of While

The standard program model in static analysis : the *control flow graph*.

The graph model used here :

- the nodes are program points $k \in \mathbb{P}$,
- the edges are labeled with *basic instructions*

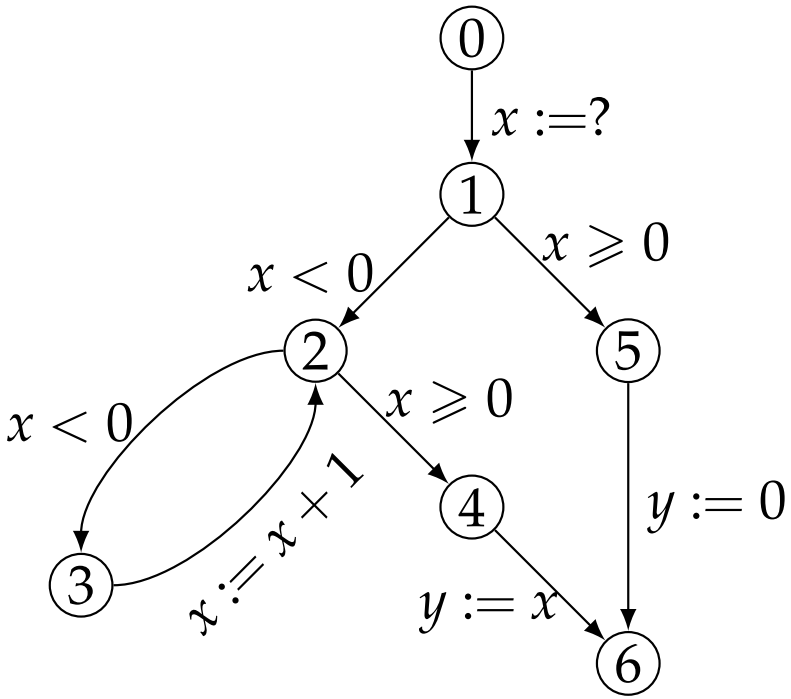
$$\begin{array}{rcll} \mathit{Instr} ::= & x := \mathit{Exp} & & \textit{assignment} \\ | & \texttt{assume}\ \mathit{test} & & \textit{execution continues only if} \\ & & & \textit{the test succeds} \end{array}$$

- formally, a cfg is a triple $(k_{\text{init}}, S, k_{\text{end}})$ with
  - $k_{\text{init}} \in \mathbb{P}$: the entry point,
  - $k_{\text{end}} \in \mathbb{P}$: the exit point,
  - $S \subseteq \mathbb{P} \times \mathit{Instr} \times \mathbb{P}$ the set of edges.

Remark : data-flow analyses are generally based on other versions of control flow graphs (instructions are put in nodes).

While syntax : example

$[^0[x :=?];$
`if` $^1[x < 0]\ \{$
  `while` $^2[x < 0]\ \{$
    $^3[x := x + 1];$
  $\};$
  $^4[y := x];$
$\}$ `else` $\{$
  $^5[y := 0];$
$\};]^6$



assume is left implicit

Small-step semantics of cfg

We first define the semantics of instructions : $\xrightarrow{i} \subseteq Env \times Env$

$$\frac{v \in \mathcal{A}[\![a]\!]\rho}{\rho \xrightarrow{x := a} \rho[x \mapsto v]} \qquad \frac{\mathbf{tt} \in \mathcal{B}[\![t]\!]\rho}{\rho \xrightarrow{\texttt{assume } t} \rho}$$

Then, a small-step relation $\rightarrow_{cfg} \subseteq State \times State$ for a $cfg = (k_{\text{init}}, S, k_{\text{end}})$

$$\frac{(k_1, i, k_2) \in S \qquad \rho_1 \xrightarrow{i} \rho_2}{(k_1, \rho_1) \rightarrow_{cfg} (k_2, \rho_2)}$$

Reachable states for control flow graphs

$$[\![(k_{\text{init}}, S, k_{\text{end}})]\!]_{\text{CFG}} = \{ (k, \rho) \mid \exists \rho_0 \in Env,\ (k_{\text{init}}, \rho_0) \rightarrow^*_{(k_{\text{init}}, S, k_{\text{end}})} (k, \rho) \}$$

Control flow graph generation (1/2)

$cfg_l(S)$ computes the edges of the control flow graph of S using l as final label.

$$
\begin{aligned}
cfg_l &\in Stm \to \mathcal{P}(\mathbb{P} \times Instr \times \mathbb{P}), \quad l \in \mathbb{P} \\
cfg_{l'}\left({}^{l}[x := e]\right) &= \{(l, x := e, l')\} \\
cfg_{l'}\left({}^{l}[\texttt{skip}]\right) &= \{(l, \texttt{assume}\ T, l')\} \qquad \text{with } T \equiv 0 = 0 \\
cfg_{l'}\left(\texttt{if}\ {}^{l}[t]\ \{\ S_1\ \}\{\ S_2\ \}\right) &= \{(l, \texttt{assume}\ t, entry(S_1))\} \cup \\
&\quad \{(l, \texttt{assume}\ neg(t), entry(S_2))\} \cup cfg_{l'}(S_1) \cup cfg_{l'}(S_2) \\
cfg_{l'}\left(\texttt{while}\ {}^{l}[t]\ \{\ S\ \}\right) &= \{(l, \texttt{assume}\ t, entry(S))\} \cup \\
&\quad cfg_l(S) \cup \{(l, \texttt{assume}\ neg(t), l')\} \\
cfg_{l'}\left(S_1;\ S_2\right) &= cfg_{entry(S_2)}(S_1) \cup cfg_{l'}(S_2)
\end{aligned}
$$

$$
\begin{aligned}
cfg &\in Prog \to \mathbb{P} \times \mathcal{P}(\mathbb{P} \times Instr \times \mathbb{P}) \times \mathbb{P} \\
cfg([P]^{end}) &= (entry(P), cfg_{end}(P), end)
\end{aligned}
$$

Control flow graph generation (2/2)

Test negation :

$$
\begin{aligned}
neg(e_1 = e_2) &= e_1 \neq e_2 \\
neg(e_1 \neq e_2) &= e_1 = e_2 \\
neg(e_1 < e_2) &= e_2 \leqslant e_1 \\
neg(e_1 \leqslant e_2) &= e_2 < e_1 \\
neg(t_1 \textbf{ and } t_2) &= neg(t_1) \textbf{ or } neg(t_1) \\
neg(t_1 \textbf{ or } t_2) &= neg(t_1) \textbf{ and } neg(t_1)
\end{aligned}
$$

Correctness of *cfg*

Theorem

For all program p,

$$[\![cfg(p)]\!]_{CFG} = [\![p]\!]_{SOS}$$

From now on, we write $[\![p]\!]$ instead of $[\![cfg(p)]\!]_{\text{CFG}}$ and we identify $cfg(p)$ and p.

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Collecting Semantics

We define a collecting semantics that gives us the set of reachable states $[\![p]\!]_k^{\text{col}}$ at each program point k.

$$\forall k \in \mathbb{P},\ [\![p]\!]_k^{\text{col}} = \{\ \rho \mid (k, \rho) \in [\![p]\!]\ \}$$

Theorem

$[\![p]\!]^{col}$ *is the least fixpoint of the following equation system.*

$$\forall k \in \mathit{labels}(p),\ X_k = X_k^{init} \cup \bigcup_{(k',i,k)\in p} [\![i]\!]\,(X_{k'})$$

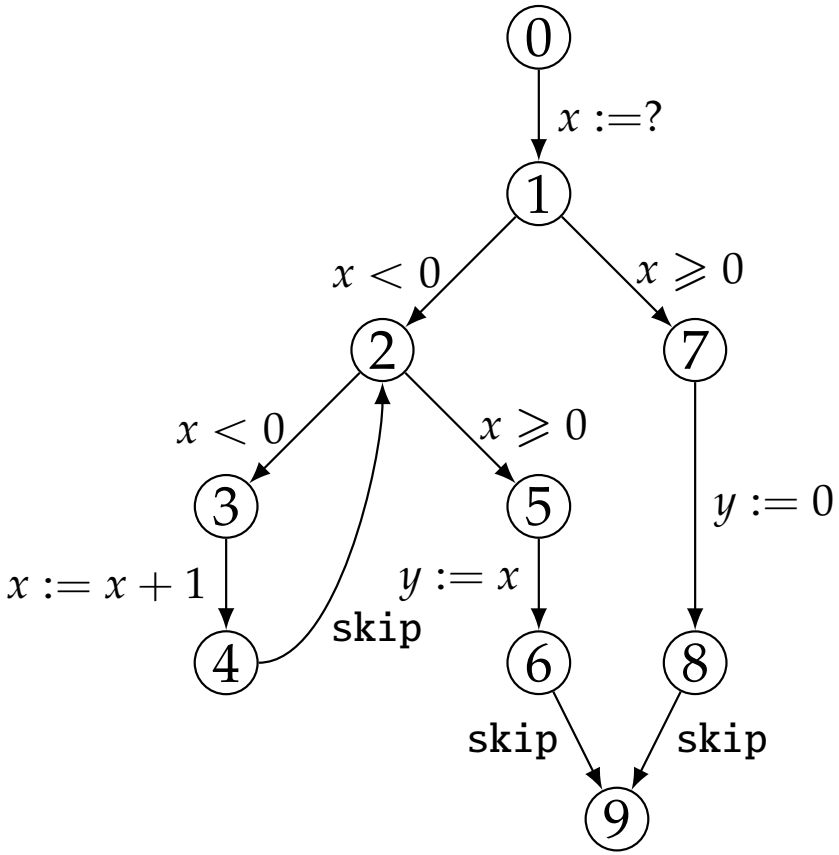
with $X_k^{\text{init}} = \begin{cases} Env & \text{if } k = k_{\text{init}} \\ \emptyset & \text{otherwise} \end{cases}$

and $\forall i \in \mathit{Instr},\ \forall X \subseteq \mathit{Env}$,

$$[\![i]\!]\,(X) = \left\{\ \rho_2 \mid \exists \rho_1 \in X,\ \rho_1 \xrightarrow{i} \rho_2\ \right\}$$

Example

For the following program, $[\![P]\!]^{\text{col}}$ is the least solution of the following equation system :



$$
\begin{aligned}
X_0 &= Env \\
X_1 &= [\![x :=?]\!](X_0) \\
X_2 &= [\![x < 0]\!](X_1) \cup X_4 \\
X_3 &= [\![x < 0]\!](X_2) \\
X_4 &= [\![x := x + 1]\!](X_3) \\
X_5 &= [\![x \geqslant 0]\!](X_2) \\
X_6 &= [\![y := x]\!](X_5) \\
X_7 &= [\![x \geqslant 0]\!](X_1) \\
X_8 &= [\![y := 0]\!](X_7) \\
X_9 &= X_6 \cup X_8
\end{aligned}
$$

Collecting semantics and exact analysis

The $(X_k)_{i=1..N}$ are specified as the least solution of a system of equations :

$$X_k = F_k(X_1, X_2, \ldots, X_N) \ , \ k \in \mathit{labels}(p)$$

or, equivalently $\vec{X} = \vec{F}(\vec{X})$.

Exact analysis :

- Thanks to Knaster-Tarski, the least solution exists (complete lattice, F_k are monotone functions),
- Kleene's fixpoint theorem (F_k are continuous functions) : it is the limit of

$$X_k^0 = \emptyset \ , \ X_k^{n+1} = F_k(X_1^n, X_2^n, \ldots, X_N^n)$$

An uncomputable problem :

- Representing the X_k may be hard (infinite sets).
- The limit may not be reachable in a finite number of steps.

Approximate analysis

Exact analysis :
Least solution of $X = F(X)$ in the complete lattice $(\mathcal{P}(Env)^N, \subseteq, \cup, \cap)$.
Computed as the limit of $X^0 = \bot, X^{n+1} = F(X^n)$

Approximate analysis :

- Replace the concrete lattice $(\mathcal{P}(Env), \subseteq, \cup, \cap)$ by an abstract lattice $(L^\sharp, \sqsubseteq^\sharp, \sqcup^\sharp, \sqcap^\sharp)$
  - whose elements can be (efficiently) represented,
  - in which we know how to compute $\sqcup^\sharp$, $\sqcap^\sharp$, $\sqsubseteq^\sharp$, ...
- "Transpose" the equation $X = F(X)$ of $\mathcal{P}(Env)^N$ into $X = F^\sharp(X)$ over $(L^\sharp)^N$.
- Compute the limit $X^0 = \bot, X^{n+1} = F^\sharp(X^n)$

Fixpoint approximation : when $L^\sharp$ does not verify the ascending chain condition, the iterative computation may not converge in a finite number of steps (or sometimes too slowly).
In this case, we need to approximate the limit (see widening/narrowing).

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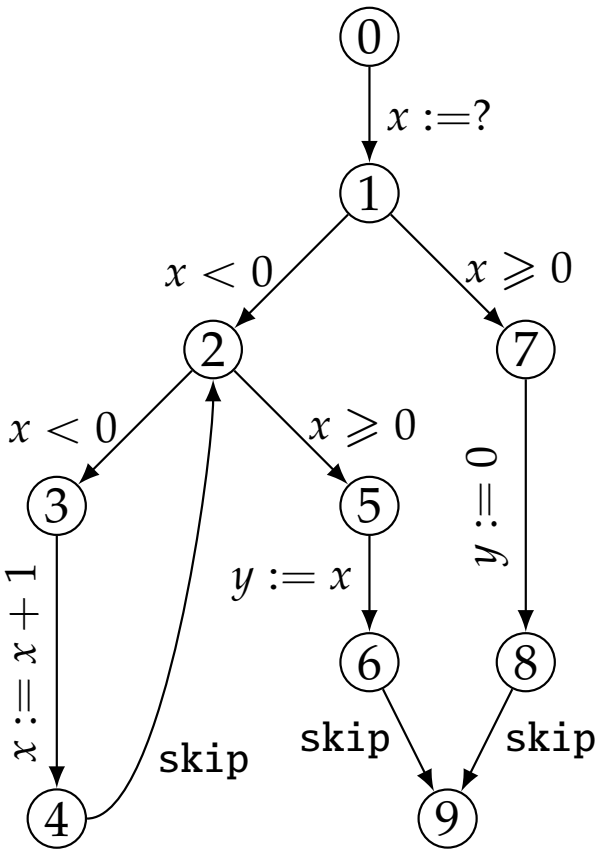
Just put some $^\sharp$...

From $\mathcal{P}(Env)$ to $Env^\sharp$

control flow graph



collecting semantics

$$
\begin{array}{lcl}
X_0 & = & Env \\
X_1 & = & [\![x :=?]\!]\,(X_0) \\
X_2 & = & [\![x < 0]\!]\,(X_1) \cup X_4 \\
X_3 & = & [\![x < 0]\!]\,(X_2) \\
X_4 & = & [\![x := x + 1]\!]\,(X_3) \\
X_5 & = & [\![x \geqslant 0]\!]\,(X_2) \\
X_6 & = & [\![y := x]\!]\,(X_5) \\
X_7 & = & [\![x \geqslant 0]\!]\,(X_1) \\
X_8 & = & [\![y := 0]\!]\,(X_7) \\
X_9 & = & X_6 \cup X_8
\end{array}
$$

abstract semantics

$$
\begin{array}{lcl}
X_0^\sharp & = & \top^\sharp_{Env} \\
X_1^\sharp & = & [\![x :=?]\!]^\sharp\,(X_0^\sharp) \\
X_2^\sharp & = & [\![x < 0]\!]^\sharp\,(X_1^\sharp) \sqcup^\sharp X_4^\sharp \\
X_3^\sharp & = & [\![x < 0]\!]^\sharp\,(X_2^\sharp) \\
X_4^\sharp & = & [\![x := x + 1]\!]^\sharp\,(X_3^\sharp) \\
X_5^\sharp & = & [\![x \geqslant 0]\!]^\sharp\,(X_2^\sharp) \\
X_6^\sharp & = & [\![y := x]\!]^\sharp\,(X_5^\sharp) \\
X_7^\sharp & = & [\![x \geqslant 0]\!]^\sharp\,(X_1^\sharp) \\
X_8^\sharp & = & [\![y := 0]\!]^\sharp\,(X_7^\sharp) \\
X_9^\sharp & = & X_6^\sharp \sqcup^\sharp X_8^\sharp
\end{array}
$$

Abstract semantics : the ingredients

- A lattice structure $(Env^\sharp, \sqsubseteq^\sharp_{Env}, \sqcup^\sharp_{Env}, \sqcap^\sharp_{Env}, \bot^\sharp_{Env}, \top^\sharp_{Env})$
  - $\sqsubseteq^\sharp_{Env}$ is an approximation of $\subseteq$
  - $\sqcup^\sharp_{Env}$ is an approximation of $\cup$
  - $\sqcap^\sharp_{Env}$ is an approximation of $\cap$
  - $\bot^\sharp_{Env}$ is an approximation of $\emptyset$
  - $\top^\sharp_{Env}$ is an approximation of Env
- For all $x \in \mathbb{V}$, $e \in Exp$, an approximation of $[\![x := e]\!]$:

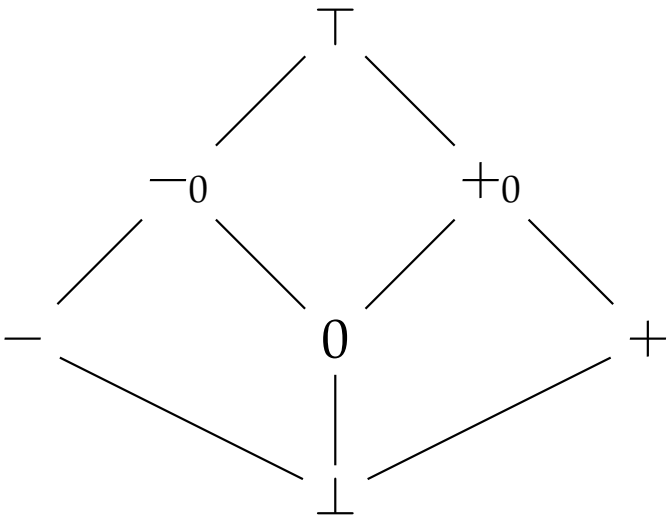
$$[\![x := e]\!]^\sharp \in Env^\sharp \to Env^\sharp$$

- For all $t \in test$, an approximation of $[\![t]\!]$:

$$[\![t]\!]^\sharp \in Env^\sharp \to Env^\sharp$$

.

An abstraction by signs



$\bot$	represents the property	$\emptyset$
$-$	represents the property	$\{ z \mid z < 0 \}$
0	represents the property	$\{0\}$
$+$	represents the property	$\{ z \mid z > 0 \}$
$-_0$	represents the property	$\{ z \mid z \leqslant 0 \}$
$+_0$	represents the property	$\{ z \mid z \geqslant 0 \}$
$\top$	represents the property	$\mathbb{Z}$

$\boldsymbol{Env}^\sharp \stackrel{\text{def}}{=} \mathbb{V} \to \boldsymbol{Sign}$: a sign is associated with each variable.

An abstraction by signs : example

$$
\begin{array}{lcl}
X_0^\sharp & = & \top_{Env}^\sharp \\
X_1^\sharp & = & [\![x :=?]\!]^\sharp\,(X_0^\sharp) \\
X_2^\sharp & = & [\![x < 0]\!]^\sharp\,(X_1^\sharp) \sqcup^\sharp X_4^\sharp \\
X_3^\sharp & = & [\![x < 0]\!]^\sharp\,(X_2^\sharp) \\
X_4^\sharp & = & [\![x := x + 1]\!]^\sharp\,(X_3^\sharp) \\
X_5^\sharp & = & [\![x \geqslant 0]\!]^\sharp\,(X_2^\sharp) \\
X_6^\sharp & = & [\![y := x]\!]^\sharp\,(X_5^\sharp) \\
X_7^\sharp & = & [\![x \geqslant 0]\!]^\sharp\,(X_1^\sharp) \\
X_8^\sharp & = & [\![y := 0]\!]^\sharp\,(X_7^\sharp) \\
X_9^\sharp & = & X_6^\sharp \sqcup^\sharp X_8^\sharp
\end{array}
\quad \xrightarrow[\text{simplifies into}]{\text{which}} \quad
\begin{array}{lcl}
X_0^\sharp & = & [x : \top;\ y : \top] \\
X_1^\sharp & = & X_0^\sharp[x \mapsto \top] \\
X_2^\sharp & = & X_1^\sharp[x \mapsto -] \sqcup^\sharp X_4^\sharp \\
X_3^\sharp & = & X_2^\sharp[x \mapsto -] \\
X_4^\sharp & = & X_3^\sharp[x \mapsto \mathrm{succ}^\sharp(X_3^\sharp(x))] \\
X_5^\sharp & = & X_2^\sharp[x \mapsto +_0] \\
X_6^\sharp & = & X_5^\sharp[y \mapsto X_5^\sharp(x)] \\
X_7^\sharp & = & X_1^\sharp[x \mapsto +_0] \\
X_8^\sharp & = & X_7^\sharp[y \mapsto 0] \\
X_9^\sharp & = & X_6^\sharp \sqcup^\sharp X_8^\sharp
\end{array}
$$

with

$$
\begin{array}{rcl}
\mathrm{succ}^\sharp(\bot) & = & \bot \\
\mathrm{succ}^\sharp(-) & = & -_0 \\
\mathrm{succ}^\sharp(0) & = & \mathrm{succ}^\sharp(+) = \mathrm{succ}^\sharp(+_0) = + \\
\mathrm{succ}^\sharp(-_0) & = & \mathrm{succ}^\sharp(\top) = \top
\end{array}
$$

Exercise

Perform a sign analysis of the program

$$
\begin{array}{l}
[^0[x := -1]; \\
\texttt{if}\ ^1[x < 0]\ \{ \\
\qquad\quad ^2[y := -3]; \\
\}\ \texttt{else}\ \{ \\
\qquad\quad ^3[y := 3]; \\
\};]^4
\end{array}
$$

- Generate the cfg.
- Generate the set of equations.
- Solve the equation system.
- Is there a loss of precision ? If yes, why ?

Outline

1. Introduction
2. Control flow graph
3. Collecting semantics
4. Approximate analysis : an informal presentation
5. Abstraction by intervals
6. Widening and narrowing
7. Galois connections
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The lattice of intervals

Elements :

$$\text{Int} \stackrel{\text{def}}{=} \{\, [a,b] \mid a,b \in \overline{\mathbb{Z}},\ a \leqslant b \,\} \cup \{\bot\} \quad \text{with } \overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$$

Order :

$$\frac{I \in \text{Int}}{\bot \sqsubseteq_{\text{Int}} I} \qquad \frac{c \leqslant a \quad b \leqslant d \quad a,b,c,d \in \overline{\mathbb{Z}}}{[a,b] \sqsubseteq_{\text{Int}} [c,d]}$$

Lattice operations :

$$\begin{aligned} I \sqcup_{\text{Int}} \bot &\stackrel{\text{def}}{=} I,\ \forall I \in \text{Int} \\ \bot \sqcup_{\text{Int}} I &\stackrel{\text{def}}{=} I,\ \forall I \in \text{Int} \\ [a,b] \sqcup_{\text{Int}} [c,d] &\stackrel{\text{def}}{=} [\min(a,c), \max(b,d)] \end{aligned}$$

$$\begin{aligned} I \sqcap_{\text{Int}} \bot &\stackrel{\text{def}}{=} \bot,\ \forall I \in \text{Int} \\ \bot \sqcap_{\text{Int}} I &\stackrel{\text{def}}{=} \bot,\ \forall I \in \text{Int} \\ [a,b] \sqcap_{\text{Int}} [c,d] &\stackrel{\text{def}}{=} \rho_{\text{Int}}([\max(a,c), \min(b,d)]) \end{aligned}$$

Normalizer : $\rho_{\text{Int}} \in (\overline{\mathbb{Z}} \times \overline{\mathbb{Z}}) \to \text{Int}$ defined by

$$\rho_{\text{Int}}(a,b) = \begin{cases} [a,b] & \text{if } a \leqslant b, \\ \bot & \text{otherwise} \end{cases}$$

Least and greatest element :

$$\begin{array}{lcl} \bot_{\text{Int}} & \stackrel{\text{def}}{=} & \bot \\ \top_{\text{Int}} & \stackrel{\text{def}}{=} & [-\infty, +\infty] \end{array}$$

Abstraction and concretisation :

$$\begin{array}{lcll} \alpha_{\text{Int}}(S) & \stackrel{\text{def}}{=} & \bot & \text{if } S = \emptyset \\ \alpha_{\text{Int}}(S) & \stackrel{\text{def}}{=} & [\min(S), \max(S)] & \text{otherwise} \end{array}$$

$$\begin{array}{lcl} \gamma_{\text{Int}}(\bot) & \stackrel{\text{def}}{=} & \emptyset \\ \gamma_{\text{Int}}([a,b]) & \stackrel{\text{def}}{=} & \{\, z \in \mathbb{Z} \mid a \leqslant z \text{ and } z \leqslant b \,\} \end{array}$$

Abstraction of basic functions

All the other operators are *strict* : they return $\bot$ if one of their arguments is $\bot$.

$$
\begin{aligned}
+^\sharp([a,b],[c,d]) &= [a+c, b+d] \\
-^\sharp([a,b],[c,d]) &= [a-d, b-c] \\
\times^\sharp([a,b],[c,d]) &= [\min(ac,ad,bc,bd), \max(ac,ad,bc,bd)]
\end{aligned}
$$

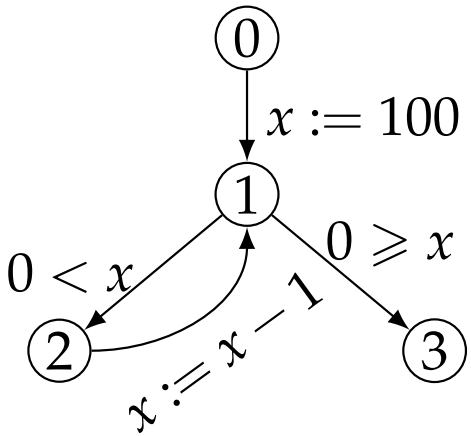
$$
\begin{aligned}
[\![=]\!]^\sharp_{\text{comp}}([a,b],[c,d]) &= ([a,b] \sqcap_{\text{Int}} [c,d], [a,b] \sqcap_{\text{Int}} [c,d]) \\
[\![<]\!]^\sharp_{\text{comp}}([a,b],[c,d]) &= ([a,b] \sqcap_{\text{Int}} [-\infty, d-1], [a+1, +\infty] \sqcap_{\text{Int}} [c,d]) \\
[\![\leqslant]\!]^\sharp_{\text{comp}}([a,b],[c,d]) &= ([a,b] \sqcap_{\text{Int}} [-\infty, d], [a, +\infty] \sqcap_{\text{Int}} [c,d]) \\
[\![\neq]\!]^\sharp_{\text{comp}}([a,b],[c,d]) &= ? \textit{ exercise...} \\
\text{const}(n)^\sharp &= [n,n]
\end{aligned}
$$

Example

```
x := 100;

while 0 < x {

   x := x − 1;
}
```



$$X_1 = [100, 100] \sqcup_{\text{Int}} \left(X_2 -^{\sharp} [1, 1]\right)$$

$$X_2 = [1, +\infty] \sqcap_{\text{Int}} X_1$$

$$X_3 = [-\infty, 0] \sqcap_{\text{Int}} X_1$$

Example : fixpoint iteration

$$
\begin{aligned}
X_1 &= [100, 100] \sqcup_{\text{Int}} \left(X_2 -^\sharp [1, 1]\right) \\
X_2 &= [1, +\infty] \sqcap_{\text{Int}} X_1 \\
X_3 &= [-\infty, 0] \sqcap_{\text{Int}} X_1
\end{aligned}
$$

Iteration strategy : $1 \to 2 \to 3 \to 1 \to 2 \to \cdots$

$$
\begin{array}{ll}
X_1^0 = \bot & X_1^{n+1} = [100, 100] \sqcup_{\text{Int}} \left(X_2^n -^\sharp [1, 1]\right) \\
X_2^0 = \bot & X_2^{n+1} = [1, +\infty] \sqcap_{\text{Int}} X_1^{n+1} \\
X_3^0 = \bot & X_3^{n+1} = [-\infty, 0] \sqcap_{\text{Int}} X_1^{n+1}
\end{array}
$$

X_1	$\bot$	$\cdots$
X_2	$\bot$	$\cdots$
X_3	$\bot$	$\cdots$

Example : fixpoint iteration

$$
\begin{aligned}
X_1 &= [100,100] \sqcup_{\text{Int}} \left(X_2 -^\sharp [1,1]\right) \\
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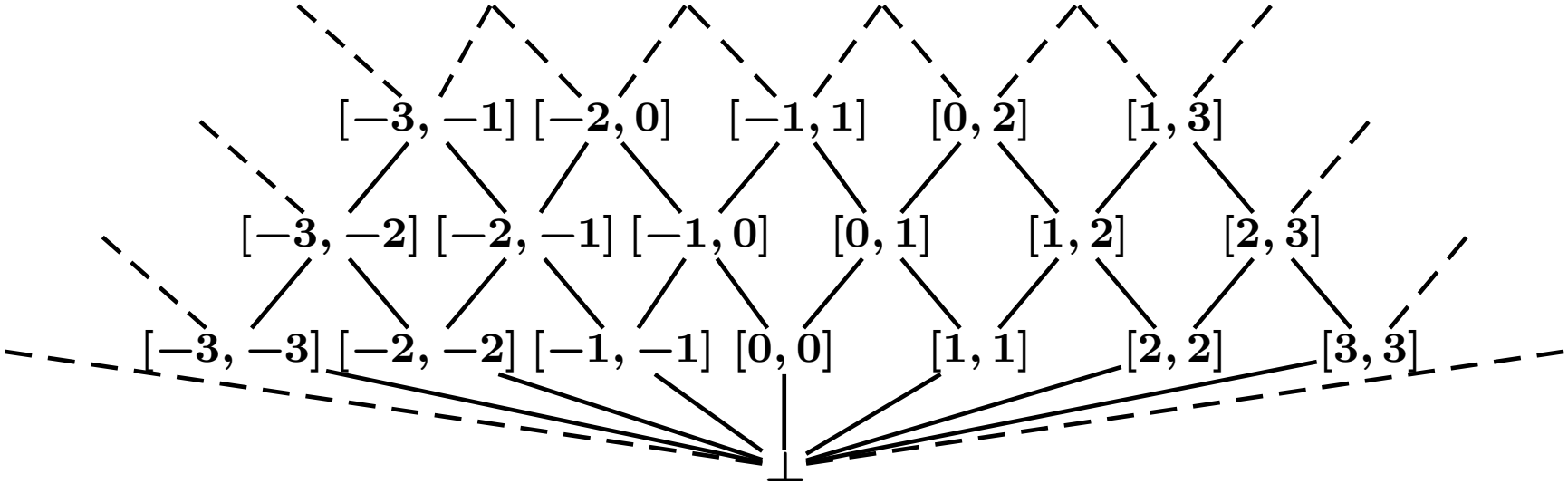
X_3 &= [-\infty,0] \sqcap_{\text{Int}} X_1
\end{aligned}
$$

Iteration strategy : $1 \to 2 \to 3 \to 1 \to 2 \to \cdots$

$$
\begin{aligned}
X_1^0 &= \bot \quad & X_1^{n+1} &= [100,100] \sqcup_{\text{Int}} \left(X_2^n -^\sharp [1,1]\right) \\
X_2^0 &= \bot \quad & X_2^{n+1} &= [1,+\infty] \sqcap_{\text{Int}} X_1^{n+1} \\
X_3^0 &= \bot \quad & X_3^{n+1} &= [-\infty,0] \sqcap_{\text{Int}} X_1^{n+1}
\end{aligned}
$$

X_1	$\bot$	$[100,100]$	$[99,100]$	$[98,100]$	$[97,100]$	$\cdots$	$[1,100]$	$[0,100]$
X_2	$\bot$	$[100,100]$	$[99,100]$	$[98,100]$	$[97,100]$	$\cdots$	$[1,100]$	$[1,100]$
X_3	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\cdots$	$\bot$	$[0,0]$

Convergence problem



The lattice of intervals does not satisfy the ascending chain condition.

Example of infinite increasing chain :

$$\bot \sqsubset [0,0] \sqsubset [0,1] \sqsubset \cdots \sqsubset [0,n] \sqsubset \cdots$$

Solution : dynamic approximation

- we extrapolate the limit thanks to a widening operator ∇

$$\bot \sqsubset [0,0] \sqsubset [0,1] \sqsubset [0,2] \sqsubset [0,+\infty] = [0,2] \nabla [0,3]$$

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Fixpoint approximation

Lemma

Let $(A, \sqsubseteq, \sqcup, \sqcap)$ be a complete lattice and f a monotone operator on A.
If a is a post-fixpoint of f (i.e. $f(a) \sqsubseteq a$), then $\mathrm{lfp}(f) \sqsubseteq a$.

We may want to over-approximate $\mathrm{lfp}(f)$ in the following cases :

- The lattice does not satisfies the ascending chain condition, the iteration $\bot, f(\bot), \ldots, f^n(\bot), \ldots$ may never terminate.
- The ascending chain condition is satisfied but the iteration chain is too long to allow an efficient computation.
- The underlying lattice is not complete, so the limits of the ascending iterations do not necessarily belong to the abstraction domain.

Widening

Idea : the standard iteration is of the form

$$x^0 = \bot, \quad x^{n+1} = F(x^n) = x^n \sqcup F(x^n)$$

We will replace it by something of the form

$$y^0 = \bot, \quad y^{n+1} = y^n \nabla F(y^n)$$

such that

(i) (y^n) is increasing,

(ii) $x^n \sqsubseteq y^n$, for all n,

(iii) and (y^n) stabilizes after a finite number of steps.

But we also want a ∇ operator that is independent of F.

Widening : definition

A widening is an operator $\nabla : L \times L \to L$ such that

- $\forall x, x' \in L, x \sqcup x' \sqsubseteq x \nabla x'$ (implies (i) & (ii))
- If $x^0 \sqsubseteq x^1 \sqsubseteq \ldots$ is an increasing chain, then the increasing chain $y^0 = x^0, y^{n+1} = y^n \nabla x^{n+1}$ stabilizes after a finite number of steps (implies (iii)).

Usage : we replace $x^0 = \bot, x^{n+1} = F(x^n)$
by $y^0 = \bot, y^{n+1} = y^n \nabla F(y^n)$

Widening : theorem

Theorem

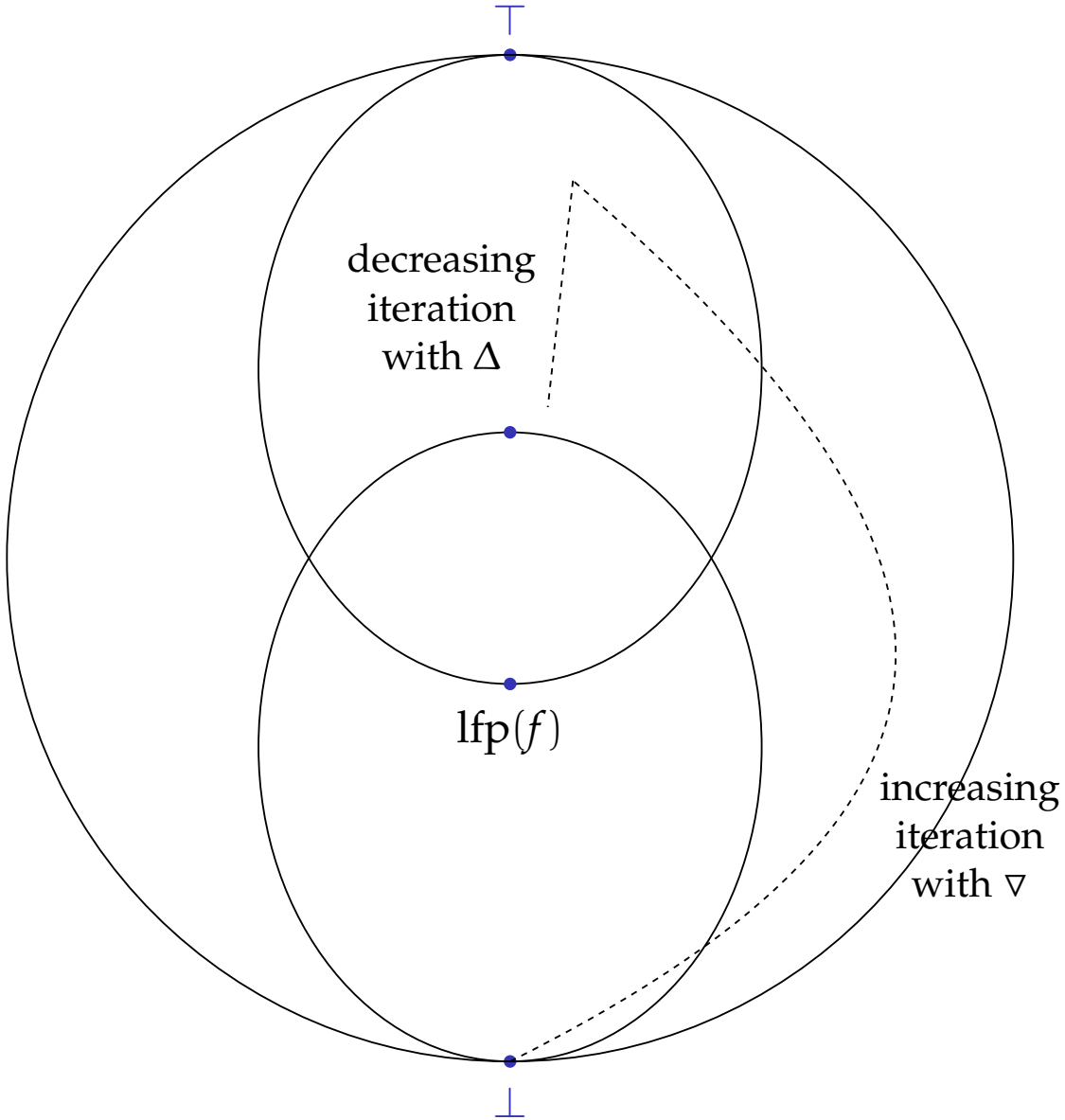
Let

- L *be a complete lattice,*
- $F : L \to L$ *be a monotone function and*
- $\nabla : L \times L \to L$ *a widening operator.*

Then, the chain $y^0 = \bot, y^{n+1} = y^n \nabla F(y^n)$ *stabilizes after a finite number of steps at a post-fixpoint* y *of* F.

Corollary : $\mathrm{lfp}(F) \sqsubseteq y$.

Scheme


⊤
decreasing
iteration
with Δ
lfp(f)
increasing
iteration
with ∇
⊥

Example : widening on intervals

Idea : as soon as a bound is not stable, we extrapolate it by $+\infty$ (or $-\infty$). After such an extrapolation, the bound can't move any more.

Definition :

$$\begin{array}{lcl} [a,b]\nabla_{\text{Int}}[a',b'] & = & [\quad \text{if } a' < a \text{ then } -\infty \text{ else } a, \\ & & \quad\ \text{if } b' > b \text{ then } +\infty \text{ else } b\,] \\ \bot\nabla_{\text{Int}}[a',b'] & = & [a',b'] \\ I\ \nabla_{\text{Int}}\ \bot & = & I \end{array}$$

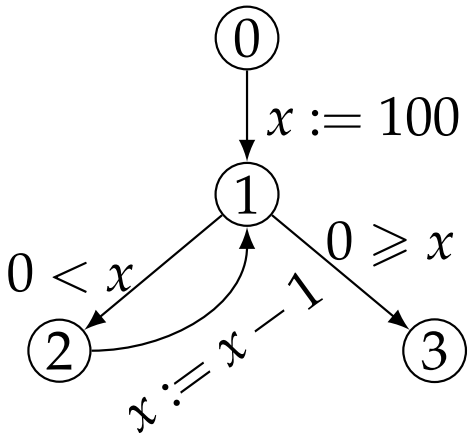
Examples :
$[-3,4]\nabla_{\text{Int}}[-3,2] = [-3,4]$
$[-3,4]\nabla_{\text{Int}}[-3,5] = [-3,+\infty]$

Example

```
x := 100;

while 0 < x {

   x := x − 1;
}
```



$$X_1 = [100, 100] \sqcup_{\text{Int}} \left(X_2 -^{\sharp} [1, 1]\right)$$

$$X_2 = [1, +\infty] \sqcap_{\text{Int}} X_1$$

$$X_3 = [-\infty, 0] \sqcap_{\text{Int}} X_1$$

Example : without widening

$$
\begin{aligned}
X_1 &= [100, 100] \sqcup_{\text{Int}} \left(X_2 -^\sharp [1, 1]\right) \\
X_2 &= [1, +\infty] \sqcap_{\text{Int}} X_1 \\
X_3 &= [-\infty, 0] \sqcap_{\text{Int}} X_1
\end{aligned}
$$

Iteration strategy : $1 \to 2 \to 3 \to 1 \to 2 \to \cdots$

$$
\begin{array}{ll}
X_1^0 = \bot & X_1^{n+1} = [100, 100] \sqcup_{\text{Int}} \left(X_2^n -^\sharp [1, 1]\right) \\
X_2^0 = \bot & X_2^{n+1} = [1, +\infty] \sqcap_{\text{Int}} X_1^{n+1} \\
X_3^0 = \bot & X_3^{n+1} = [-\infty, 0] \sqcap_{\text{Int}} X_1^{n+1}
\end{array}
$$

X_1	$\bot$	$\cdots$
X_2	$\bot$	$\cdots$
X_3	$\bot$	$\cdots$

Example : without widening

$$
\begin{aligned}
X_1 &= [100,100] \sqcup_{\text{Int}} \left(X_2 -^\sharp [1,1]\right) \\
X_2 &= [1,+\infty] \sqcap_{\text{Int}} X_1 \\
X_3 &= [-\infty,0] \sqcap_{\text{Int}} X_1
\end{aligned}
$$

Iteration strategy : $1 \to 2 \to 3 \to 1 \to 2 \to \cdots$

$$
\begin{aligned}
X_1^0 &= \bot \quad & X_1^{n+1} &= [100,100] \sqcup_{\text{Int}} \left(X_2^n -^\sharp [1,1]\right) \\
X_2^0 &= \bot \quad & X_2^{n+1} &= [1,+\infty] \sqcap_{\text{Int}} X_1^{n+1} \\
X_3^0 &= \bot \quad & X_3^{n+1} &= [-\infty,0] \sqcap_{\text{Int}} X_1^{n+1}
\end{aligned}
$$

X_1	$\bot$	$[100,100]$	$[99,100]$	$[98,100]$	$[97,100]$	$\cdots$	$[1,100]$	$[0,100]$
X_2	$\bot$	$[100,100]$	$[99,100]$	$[98,100]$	$[97,100]$	$\cdots$	$[1,100]$	$[1,100]$
X_3	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\cdots$	$\bot$	$[0,0]$

Example : with widening at each node of the cfg

$$
\begin{aligned}
X_1 &= [100,100] \sqcup_{\text{Int}} \left(X_2 -^\sharp [1,1]\right) \\
X_2 &= [1,+\infty] \sqcap_{\text{Int}} X_1 \\
X_3 &= [-\infty,0] \sqcap_{\text{Int}} X_1
\end{aligned}
$$

Iteration strategy : $1 \to 2 \to 3 \to 1 \to 2 \to \cdots$

$$
\begin{array}{ll}
X_1^0 = \bot & X_1^{n+1} = X_1^n \nabla_{\text{Int}} \left([100,100] \sqcup_{\text{Int}} \left(X_2^n -^\sharp [1,1]\right)\right) \\
X_2^0 = \bot & X_2^{n+1} = X_2^n \nabla_{\text{Int}} \left([1,+\infty] \sqcap_{\text{Int}} X_1^{n+1}\right) \\
X_3^0 = \bot & X_3^{n+1} = X_3^n \nabla_{\text{Int}} \left([-\infty,0] \sqcap_{\text{Int}} X_1^{n+1}\right)
\end{array}
$$

X_1	$\bot$
X_2	$\bot$
X_3	$\bot$

Example : with widening at each node of the cfg

$$
\begin{aligned}
X_1 &= [100,100] \sqcup_{\text{Int}} \left(X_2 -^\sharp [1,1]\right) \\
X_2 &= [1,+\infty] \sqcap_{\text{Int}} X_1 \\
X_3 &= [-\infty,0] \sqcap_{\text{Int}} X_1
\end{aligned}
$$

Iteration strategy : $1 \to 2 \to 3 \to 1 \to 2 \to \cdots$

$$
\begin{array}{ll}
X_1^0 = \bot & X_1^{n+1} = X_1^n \nabla_{\text{Int}} \left([100,100] \sqcup_{\text{Int}} \left(X_2^n -^\sharp [1,1]\right)\right) \\
X_2^0 = \bot & X_2^{n+1} = X_2^n \nabla_{\text{Int}} \left([1,+\infty] \sqcap_{\text{Int}} X_1^{n+1}\right) \\
X_3^0 = \bot & X_3^{n+1} = X_3^n \nabla_{\text{Int}} \left([-\infty,0] \sqcap_{\text{Int}} X_1^{n+1}\right)
\end{array}
$$

X_1	$\bot$	$[100,100]$	$[-\infty,100]$
X_2	$\bot$	$[100,100]$	$[-\infty,100]$
X_3	$\bot$	$\bot$	$[-\infty,0]$

Improving fixpoint approximation

Idea : iterating a little more may help...

Theorem

Let $(A, \sqsubseteq, \sqcup, \sqcap)$ be a complete lattice, f a monotone operator on A and a a post-fixpoint of f.

The chain $(x_n)_n$ defined by

$$\begin{cases} x_0 & = & a \\ x_{k+1} & = & f(x_k) \end{cases}$$

admits for limit ($\bigsqcap \{x_n\}$) the greatest fixpoint of f smaller than a (written $\mathrm{gfp}_a(f)$).

In particular,

- $\mathrm{lfp}(f) \sqsubseteq \bigsqcap \{x_n\}$.
- *Each intermediate step is a correct approximation :*

$$\forall k,\ \mathrm{lfp}(f) \sqsubseteq \mathrm{gfp}_a(f) \sqsubseteq x_k \sqsubseteq a$$

Narrowing : definition

A *narrowing* is an operator $\Delta : L \times L \to L$ such that

- $\forall x, x' \in L, x' \sqsubseteq x \Delta x' \sqsubseteq x$
- If $x^0 \sqsupseteq x^1 \sqsupseteq \ldots$ is a decreasing chain, then the chain $y^0 = x^0, y^{n+1} = y^n \Delta x^{n+1}$ stabilizes after a finite number of steps.

Narrowing : decreasing iteration

Theorem

If Δ is a narrowing operator on a poset $(A, \sqsubseteq)$,
and f is a monotone operator on A
and a is a post-fixpoint of f
then the chain $(x_n)_n$ defined by

$$\begin{cases} x_0 & = & a \\ x_{k+1} & = & x_k \Delta f(x_k) \end{cases}$$

stabilizes after a finite number of steps
at a post-fixpoint of f lower than a.

Narrowing on intervals

$$
\begin{array}{rcl}
[a,b]\Delta_{\text{Int}}[c,d] & = & [\text{if } a = -\infty \text{ then } c \text{ else } a \ ; \ \text{if } b = +\infty \text{ then } d \text{ else } b] \\
I \ \Delta_{\text{Int}} \ \bot & = & \bot \\
\bot \ \Delta_{\text{Int}} \ I & = & \bot
\end{array}
$$

Intuition : we only improve infinite bounds.

In practice : a few standard iterations already improve a lot the result that has been obtained after widening...

- Assignments by constants and conditional guards make the decreasing iterations efficient : they *filter* the (too big) approximations computed by the widening

Example : with narrowing at each node of the cfg

$$
\begin{aligned}
X_1 &= [100, 100] \sqcup_{\text{Int}} \left(X_2 -^\sharp [1, 1]\right) \\
X_2 &= [1, +\infty] \sqcap_{\text{Int}} X_1 \\
X_3 &= [-\infty, 0] \sqcap_{\text{Int}} X_1
\end{aligned}
$$

Iteration strategy : $1 \to 2 \to 3 \to 1 \to 2 \to \cdots$

$$
\begin{array}{ll}
X_1^0 = [-\infty, 100] & X_1^{n+1} = X_1^n \Delta_{\text{Int}} \left([100, 100] \sqcup_{\text{Int}} \left(X_2^n -^\sharp [1, 1]\right)\right) \\
X_2^0 = [-\infty, 100] & X_2^{n+1} = X_2^n \Delta_{\text{Int}} \left([1, +\infty] \sqcap_{\text{Int}} X_1^{n+1}\right) \\
X_3^0 = [-\infty, 0] & X_3^{n+1} = X_3^n \Delta_{\text{Int}} \left([-\infty, 0] \sqcap_{\text{Int}} X_1^{n+1}\right)
\end{array}
$$

X_1	$[-\infty, 100]$
X_2	$[-\infty, 100]$
X_3	$[-\infty, 0]$

Example : with narrowing at each node of the cfg

$$
\begin{aligned}
X_1 &= [100, 100] \sqcup_{\text{Int}} \left(X_2 -^\sharp [1, 1]\right) \\
X_2 &= [1, +\infty] \sqcap_{\text{Int}} X_1 \\
X_3 &= [-\infty, 0] \sqcap_{\text{Int}} X_1
\end{aligned}
$$

Iteration strategy : $1 \to 2 \to 3 \to 1 \to 2 \to \cdots$

$$
\begin{array}{ll}
X_1^0 = [-\infty, 100] & X_1^{n+1} = X_1^n \Delta_{\text{Int}} \left([100, 100] \sqcup_{\text{Int}} \left(X_2^n -^\sharp [1, 1]\right)\right) \\
X_2^0 = [-\infty, 100] & X_2^{n+1} = X_2^n \Delta_{\text{Int}} \left([1, +\infty] \sqcap_{\text{Int}} X_1^{n+1}\right) \\
X_3^0 = [-\infty, 0] & X_3^{n+1} = X_3^n \Delta_{\text{Int}} \left([-\infty, 0] \sqcap_{\text{Int}} X_1^{n+1}\right)
\end{array}
$$

X_1	$[-\infty, 100]$	$[-\infty, 100]$	$[0, 100]$
X_2	$[-\infty, 100]$	$[1, 100]$	$[1, 100]$
X_3	$[-\infty, 0]$	$[-\infty, 0]$	$[0, 0]$

The particular case of an equation system

Consider a system

$$\begin{cases} x_1 &= f_1(x_1, \ldots, x_n) \\ &\vdots \\ x_n &= f_n(x_1, \ldots, x_n) \end{cases}$$

with $f_1, \ldots, f_n$ monotones.

Standard iteration :

$$\begin{array}{rcl} x_1^{i+1} &=& f_1(x_1^i, \ldots, x_n^i) \\ x_2^{i+1} &=& f_2(x_1^i, \ldots, x_n^i) \\ &\vdots& \\ x_n^{i+1} &=& f_n(x_1^i, \ldots, x_n^i) \end{array}$$

Standard iteration with widening :

$$\begin{array}{rcl} x_1^{i+1} &=& x_1^i \nabla f_1(x_1^i, \ldots, x_n^i) \\ x_2^{i+1} &=& x_2^i \nabla f_2(x_1^i, \ldots, x_n^i) \\ &\vdots& \\ x_n^{i+1} &=& x_n^i \nabla f_n(x_1^i, \ldots, x_n^i) \end{array}$$

The particular case of an equation system

$$\begin{cases} x_1 &= f_1(x_1, \ldots, x_n) \\ &\vdots \\ x_n &= f_n(x_1, \ldots, x_n) \end{cases}$$

It is sufficient (and generally more precise) to use ∇ for a selection of index W provided that each dependence cycle in the system goes through at least one point in W.

$$\forall k = 1..n,\ x_k^{i+1} = \begin{array}{ll} x_k^i \nabla f_k(x_1^i, \ldots, x_n^i) & \text{if } k \in W \\ f_k(x_1^i, \ldots, x_n^i) & \text{otherwise} \end{array}$$

Chaotic iteration : at each step, we use only one equation, without forgetting one for ever.
Beware : this time the iteration strategy may affect the precision of the obtained post-fixpoint !
Delayed widening : It is generally better to wait a few standard iterations before launching the widenings.

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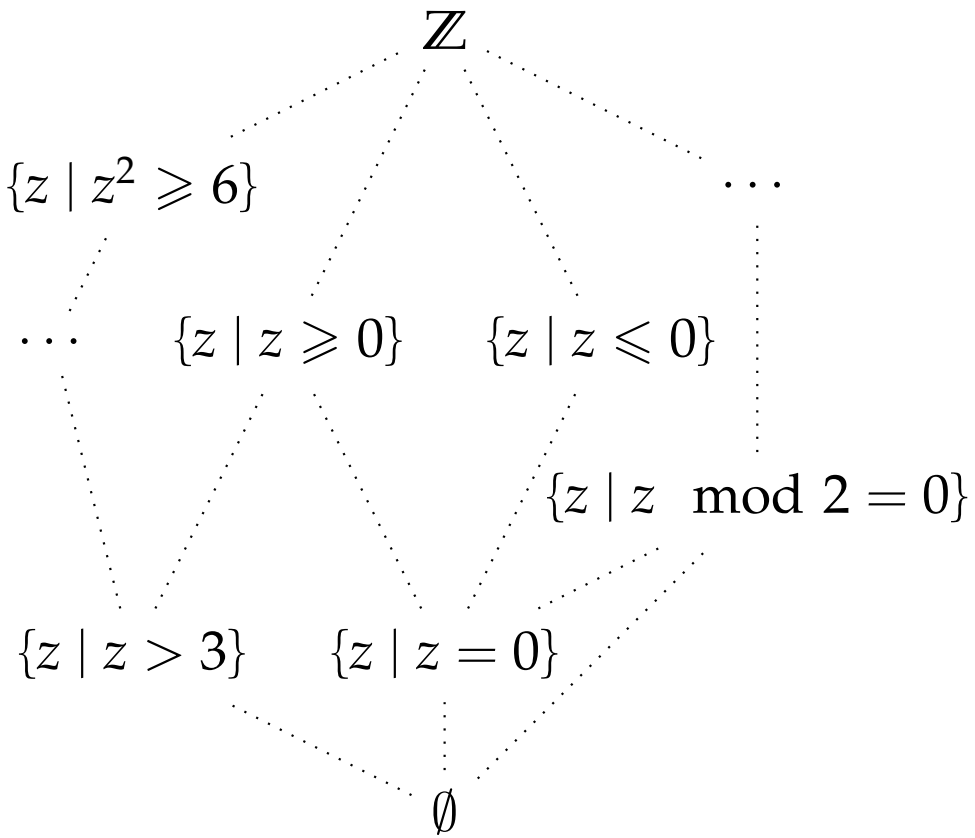
What is an approximation ?. Example : $\mathcal{P}(\mathbb{Z})$

For a program with only one variable, the set of concrete properties is

$$\mathcal{A} = \mathcal{P}(Env) = \mathcal{P}(\mathbb{Z})$$

$$= \left\{ \begin{array}{l} \{z \mid z > 3\}, \\ \{z \mid z^2 \geqslant 6\}, \\ \{z \mid z \mod 2 = 0\}, \\ \{z \mid z = 0\}, \\ \{z \mid z \geqslant 0\}, \\ \ldots \end{array} \right\}$$

which can be partially ordered by $\subseteq$ (logical implication).



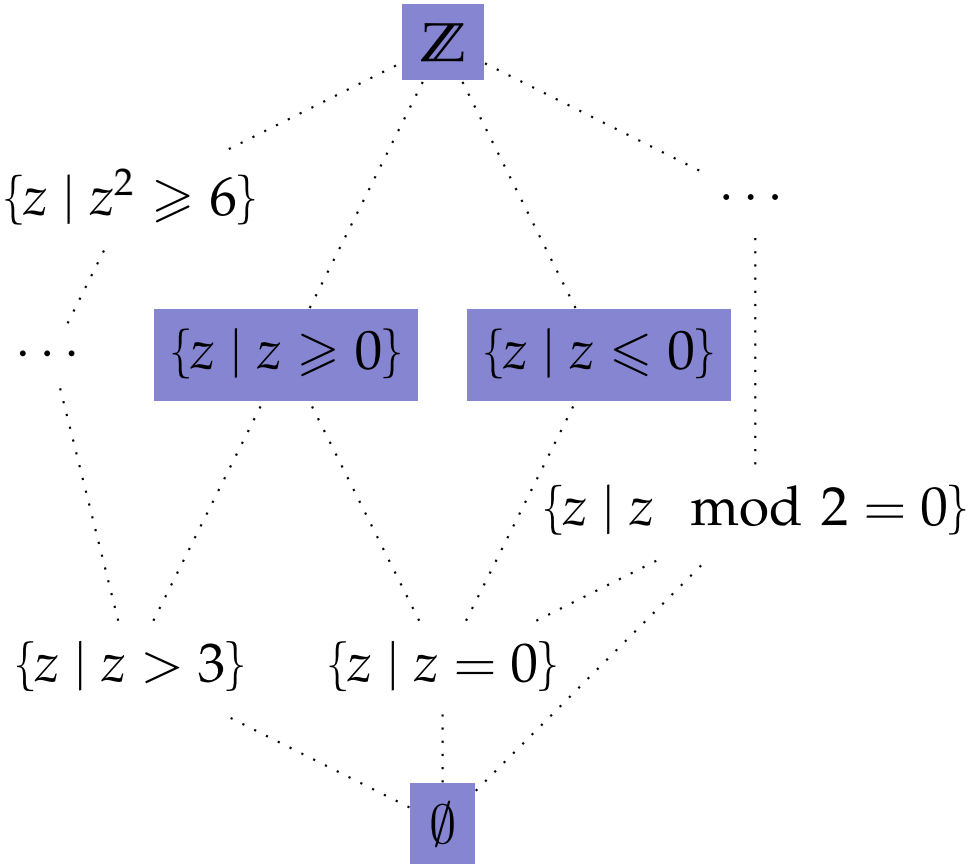
Example : signs

We select only 4 properties

$$\overline{\mathcal{A}} = \{ \emptyset, \mathbb{Z}^+, \mathbb{Z}^-, \mathbb{Z} \}$$

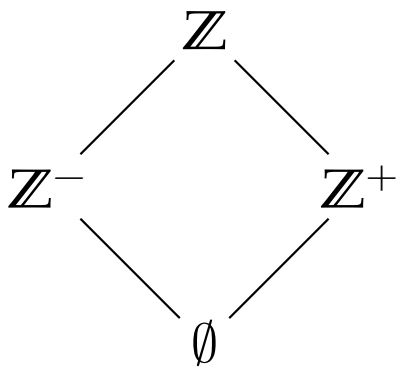
The property $\ll$ to be greater than 3 $\gg$ (*i.e.* $p_3\{ z \in \mathbb{Z} \mid z > 3 \}$) is correctly approximated by $\mathbb{Z}^+$ et $\mathbb{Z}$.

$$p_3 \subseteq \mathbb{Z}^+ \qquad p_3 \subseteq \mathbb{Z}$$



Example : sign analysis

$\mathcal{A} = \mathcal{P}(\mathbb{Z})$ $\quad$ $\overline{\mathcal{A}} = \{\emptyset, \mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}^-\}$



The successor operation $x \mapsto x + 1$ is approximated by

$\overline{p}$	$\emptyset$	$\mathbb{Z}$	$\mathbb{Z}^+$	$\mathbb{Z}^-$
$\overline{\text{succ}}(\overline{p})$	$\emptyset$	$\mathbb{Z}$	$\mathbb{Z}^+$	$\mathbb{Z}$

The predecessor operation $x \mapsto x - 1$ is approximated by

$\overline{p}$	$\emptyset$	$\mathbb{Z}$	$\mathbb{Z}^+$	$\mathbb{Z}^-$
$\overline{\text{pred}}(\overline{p})$	$\emptyset$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^-$

Which property of $\overline{\mathcal{A}}$ do we choose to approximate the property $\{0\}$?

- if we choose $\mathbb{Z}^+$

  `x = 0; y = x +1; z = x -1` $\quad\rightarrow\quad$ $\mathtt{x} \in \mathbb{Z}^+, \mathtt{y} \in \mathbb{Z}^+, \mathtt{z} \in \mathbb{Z}$

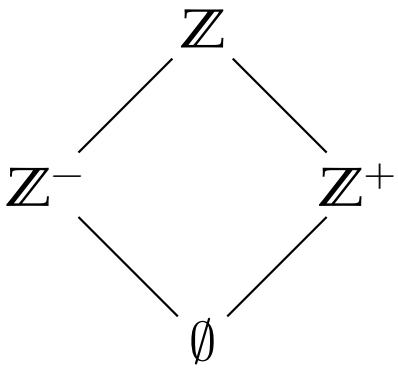
- if we choose $\mathbb{Z}^-$

  `x = 0; y = x +1; z = x -1` $\quad\rightarrow\quad$ $\mathtt{x} \in \mathbb{Z}^-, \mathtt{y} \in \mathbb{Z}, \mathtt{z} \in \mathbb{Z}^-$

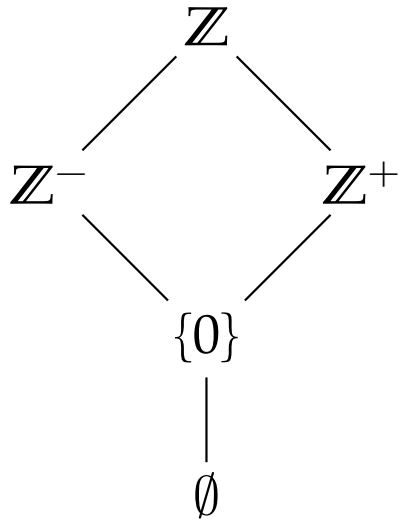
- however, we would prefer : $\mathtt{y} \in \mathbb{Z}^+, \mathtt{z} \in \mathbb{Z}^-$

Example : sign analysis

$$\mathcal{A} = \mathcal{P}(\mathbb{Z})$$



$$\overline{\mathcal{A}}_1 = \{\emptyset, \mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}^-\}$$



$$\overline{\mathcal{A}}_2 = \{\emptyset, \mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}^-, \{0\}\}$$

Problem : $\{0\}$ does not have a best approximation in $\overline{\mathcal{A}}_1$.

- because $\mathbb{Z}^- \cap \mathbb{Z}^+ \notin \overline{\mathcal{A}}_1$

But $\overline{\mathcal{A}}_2$ is closed under intersection.

- $\mathbb{Z}^- \cap \mathbb{Z}^+, \mathbb{Z} \cap \mathbb{Z}^+, \ldots, \{0\} \cap \mathbb{Z}^- \cap \mathbb{Z}^+, \ldots \in \overline{\mathcal{A}}_2$

Galois connections

Definition

Let $(L_1, \sqsubseteq_1, \sqcup_1, \sqcap_1)$ and $(L_2, \sqsubseteq_2, \sqcup_2, \sqcap_2)$ be two complete lattices.

A pair of functions $\alpha \in L_1 \to L_2$ and $\gamma \in L_2 \to L_1$ is a *Galois connection* if it verifies the condition

$$\forall x_1 \in L_1, \forall x_2 \in L_2,\ \alpha(x_1) \sqsubseteq_2 x_2 \iff x_1 \sqsubseteq_1 \gamma(x_2)$$

What is a *good* approximation space ?

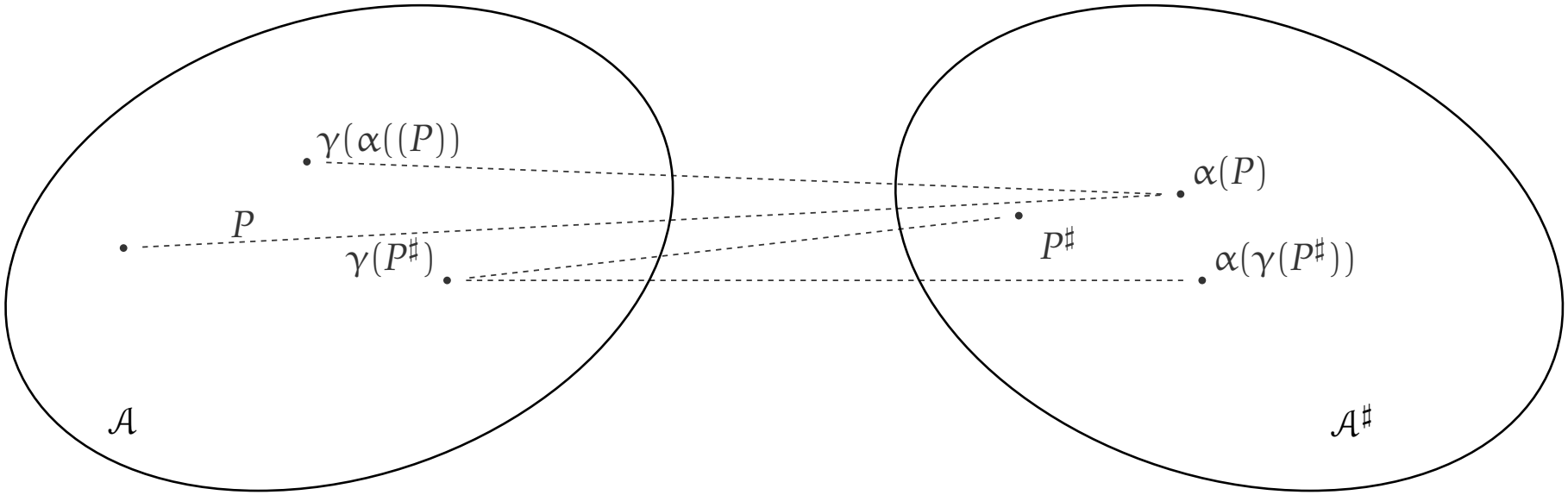
1. concrete world : a complete lattice, generally of the form $(\mathcal{P}(\mathcal{D}), \subseteq, \bigcup, \bigcap)$
2. abstract world : a complete lattice $\left(A^\sharp, \sqsubseteq^\sharp, \bigsqcup^\sharp, \bigsqcap^\sharp\right)$
3. link between them : Galois connection.

$$\left(\mathcal{P}(\mathcal{D}), \subseteq, \bigcup, \bigcap\right) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \left(A^\sharp, \sqsubseteq^\sharp, \sqcup^\sharp, \sqcap^\sharp\right)$$

$$\begin{aligned} &\ll a^\sharp \in A^\sharp \text{ is a correct approximation of } a \in \mathcal{P}(\mathcal{D}) \gg \\ \iff &\quad \alpha(a) \sqsubseteq^\sharp a^\sharp \\ \iff &\quad a \subseteq \gamma(a^\sharp) \end{aligned}$$

α : abstraction function $\qquad$ γ : concretisation function

Remark : in practice, γ is sufficient to prove the soundness of analyses, but we lose some "nice" theorems...


$\gamma(\alpha((P))$
P
$\gamma(P^\sharp)$
$\alpha(P)$
$P^\sharp$
$\alpha(\gamma(P^\sharp))$
$\mathcal{A}$
$\mathcal{A}^\sharp$

Galois connections, properties (1)

If $\alpha : L \to L^\sharp$, $\gamma : L^\sharp \to L$ is a Galois connection.

- $\gamma \circ \alpha$ is extensive : $\forall x \in L,\ x \sqsubseteq \gamma(\alpha(x))$:
  the abstraction is a correct approximation
- $\alpha \circ \gamma$ is retractive : $\forall y \in L^\sharp,\ \alpha(\gamma(y) \sqsubseteq^\sharp y$:
  if $L^\sharp$ is « well chosen », $\alpha \circ \gamma = \mathrm{id}$ and (hence) γ is injective (two distinct abstract values never represent the same concrete property)
- α and γ are monotone
  « the more information we have on a concrete object, the more we have on its abstraction, and *vice-versa* »

Remarks :

- $x \sqsubseteq x'$ means that x handles more information than x', i.e. gives a more precise information,
- $\top$ means : everything is possible.

Function approximation (1)

When some computations in the concrete world are uncomputable or too costly, the abstract world can be used to execute a simplified version of these computations.

- the abstract computation must always give a conservative answer w.r.t. the concrete computation

Let $f \in \mathcal{A} \to \mathcal{A}$ in the concrete world and $f^\sharp \in \mathcal{A}^\sharp \to \mathcal{A}^\sharp$ which correctly approximates each concrete computation.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{A} \\ \downarrow & & \downarrow \\ \mathcal{A}^\sharp & \xrightarrow{f^\sharp} & \mathcal{A}^\sharp \end{array}$$

$$\alpha(a) \sqsubseteq^\sharp a^\sharp \Rightarrow \alpha(f(a)) \sqsubseteq^\sharp f^\sharp(a^\sharp)$$

Function approximation (1)

If $\alpha : L \to L^\sharp, \gamma : L^\sharp \to L$ is a Galois connection.

Definition

A function $f^\sharp \in \mathcal{A}^\sharp \to \mathcal{A}^\sharp$ is called a *correct approximation* of $f \in \mathcal{A} \to \mathcal{A}$ if

$$\forall a \in \mathcal{A}, a^\sharp \in \mathcal{A}^\sharp, \alpha(a) \sqsubseteq^\sharp a^\sharp \Rightarrow \alpha(f(a)) \sqsubseteq^\sharp f^\sharp(a^\sharp)$$

For the monotone abstract functions, we can state several equivalent criteria.

Theorem

For a monotone function $f^\sharp \in \mathcal{A}^\sharp \to \mathcal{A}^\sharp$ and a function $f \in \mathcal{A} \to \mathcal{A}$, the four following assertions are equivalent :

(i) *$f^\sharp$ is correct approximation of f,*

(ii) $\alpha \circ f \dot{\sqsubseteq}^\sharp f^\sharp \circ \alpha$

(iii) $\alpha \circ f \circ \gamma \dot{\sqsubseteq}^\sharp f^\sharp$

(iv) $f \circ \gamma \dot{\sqsubseteq} \gamma \circ f^\sharp$

Fixpoint transfer

Theorem

Given a Galois connection $(\mathcal{A}, \sqsubseteq, \bigsqcup, \bigsqcap) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \left(\mathcal{A}^\sharp, \sqsubseteq^\sharp, \bigsqcup^\sharp, \bigsqcap^\sharp\right)$*, a monotone function* $f^\sharp \in \mathcal{A}^\sharp \to \mathcal{A}^\sharp$ *and a monotone function* $f \in \mathcal{A} \to \mathcal{A}$ *which verifies* $\alpha \circ f = f^\sharp \circ \alpha$*, we have*

$$\alpha(\mathrm{lfp}(f)) = \mathrm{lfp}(f^\sharp)$$

Theorem

Given a Galois connection $(\mathcal{A}, \sqsubseteq, \bigsqcup, \bigsqcap) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \left(\mathcal{A}^\sharp, \sqsubseteq^\sharp, \bigsqcup^\sharp, \bigsqcap^\sharp\right)$*, a monotone function* $f^\sharp \in \mathcal{A}^\sharp \to \mathcal{A}^\sharp$ *and a monotone function* $f \in \mathcal{A} \to \mathcal{A}$ *which verify* $\alpha \circ f \sqsubseteq^\sharp f^\sharp \circ \alpha$*, we have*

$$\alpha(\mathrm{lfp}(f)) \sqsubseteq^\sharp \mathrm{lfp}(f^\sharp)$$

Best abstract interpretation

From the equivalences

$$\alpha \circ f \mathrel{\dot{\sqsubseteq}^\sharp} f^\sharp \circ \alpha \iff \alpha \circ f \circ \gamma \mathrel{\dot{\sqsubseteq}^\sharp} f^\sharp \iff f \circ \gamma \mathrel{\dot{\sqsubseteq}} \gamma \circ f^\sharp$$

we have that $\alpha \circ f \circ \gamma$ is the **best** abstract function $f^\sharp$ which verifies $\alpha(\mathrm{lfp}(f)) \sqsubseteq^\sharp \mathrm{lfp}(f^\sharp)$.

But : $\alpha \circ f \circ \gamma$ must not be confused with an implementation of $f^\sharp$.

In practice,

- $\alpha \circ f \circ \gamma$ gives a **specification** of the best abstraction,
- not an **algorithm** (because α is rarely computable)

Exercise

Suppose we analyze programs with 3 variables x_1, x_2 and x_3. A state is a triple (v_1, v_2, v_3) where $v_i \in \mathbb{Z}$ is the current value of variable x_i.

$$Env = \mathbb{Z}^3 \qquad \mathcal{A} = \mathcal{P}(\mathbb{Z}^3)$$

We would like to design an abstract domain that tracks equalities between variables, and select the following properties for $\overline{\mathcal{A}}$

$$\overline{\mathcal{A}} = \left\{ \begin{array}{c} \emptyset, \\ \{(v, v, u) \mid v \in \mathbb{Z},\ u \in \mathbb{Z}\}, \\ \{(u, v, v) \mid u \in \mathbb{Z},\ v \in \mathbb{Z}\}, \\ \{(v, u, v) \mid u \in \mathbb{Z},\ v \in \mathbb{Z}\}, \\ \mathbb{Z}^3 \end{array} \right\}$$

What is the lattice (abstract domain) for this selection of properties?

We next want to define the corresponding Galois connection, but that turns out to be difficult—why? What's the problem with our abstract domain? How can it be repaired?

Exercise

Consider the following concrete operators

$$[\![x_1 := ?]\!], \quad [\![x_1 := x_2]\!], \quad [\![\texttt{assume}\ x_1 = x_2]\!] \in \mathcal{P}(\mathbb{Z}^3) \to \mathcal{P}(\mathbb{Z}^3)$$

Using the abstraction proposed in the previous exercise, give sound approximations for these operators.

Prove they are sound wrt. the given abstraction choice.

Prove they are optimal wrt. the given abstraction choice.

Outline

1 Introduction

2 Control flow graph

3 Collecting semantics

4 Approximate analysis : an informal presentation

5 Abstraction by intervals

6 Widening and narrowing

7 Galois connections

8 Polyhedral abstract interpretation

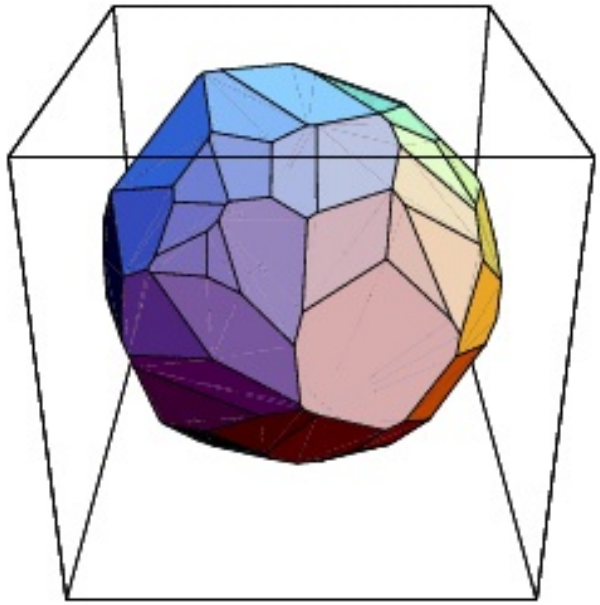
9 References

Polyhedral abstract interpretation

Polyhedral analysis seeks to discover invariants of **linear equality and inequality** relations (such as $x = y$ or $x \leqslant 2y + z$) among the variables of an imperative program.

A convex polyhedron can be defined

- algebraically as the set of solutions of a system of linear inequalities.
- geometrically, as a finite intersection of half-spaces.

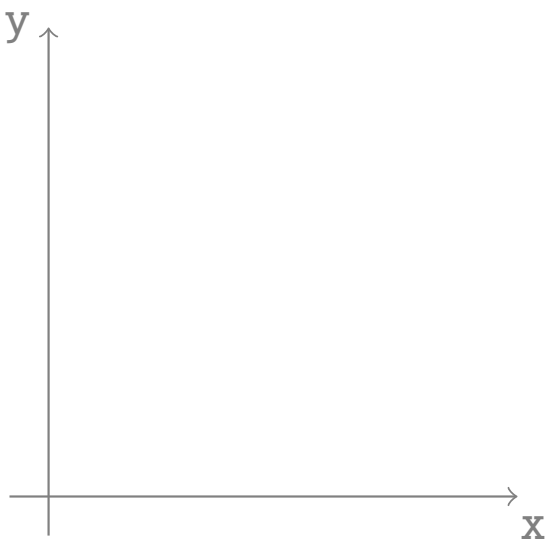


The classical reference :

> *Automatic discovery of linear restraints among variables of a program.*
> P. Cousot and N. Halbwachs. POPL'78.

Polyhedral analysis

State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.




```
x = 0; y = 0;


while (x<6) {
  if (?) {

    y = y+2;

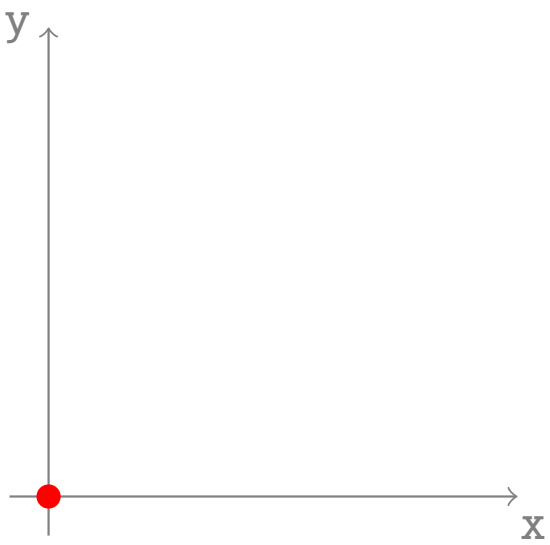
  };


  x = x+1;

}
```

Polyhedral analysis

State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.

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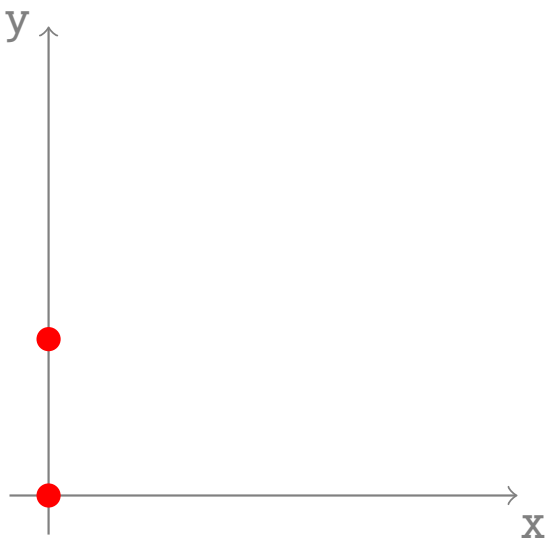

At junction points, we over-approximates union by a convex union.

```
x = 0; y = 0;
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```
while (x<6) {
  if (?) {
```
$\{x = 0 \wedge y = 0\}$

```
    y = y+2;
```
$\{x = 0 \wedge y = 2\}$

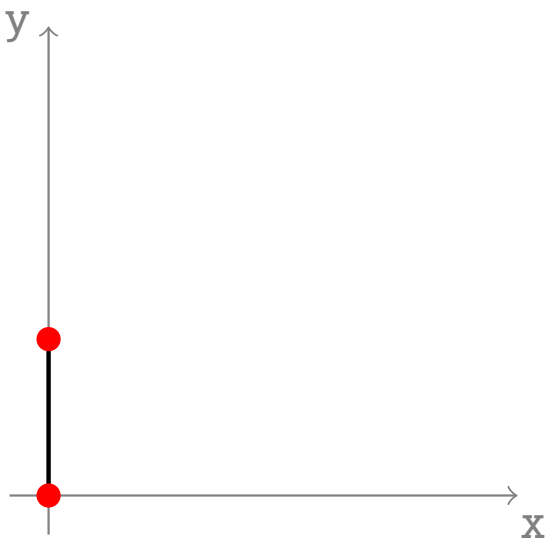
```
  };
```
$\{x = 0 \wedge y = 0\} \uplus \{x = 0 \wedge y = 2\}$

```
  x = x+1;

}
```

Polyhedral analysis

State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.



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$\{x = 0 \wedge y = 2\}$

```
  };
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$\{x = 0 \wedge 0 \leqslant y \leqslant 2\}$

```
  x = x+1;

}
```

Polyhedral analysis

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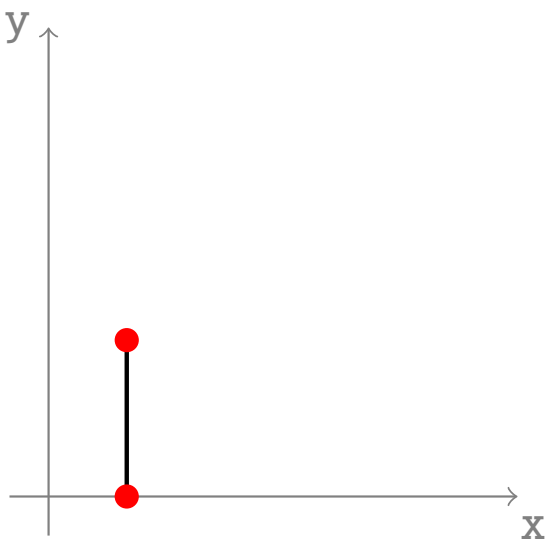


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  x = x+1;
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$\{x = 1 \wedge 0 \leqslant y \leqslant 2\}$

```
}
```

Polyhedral analysis

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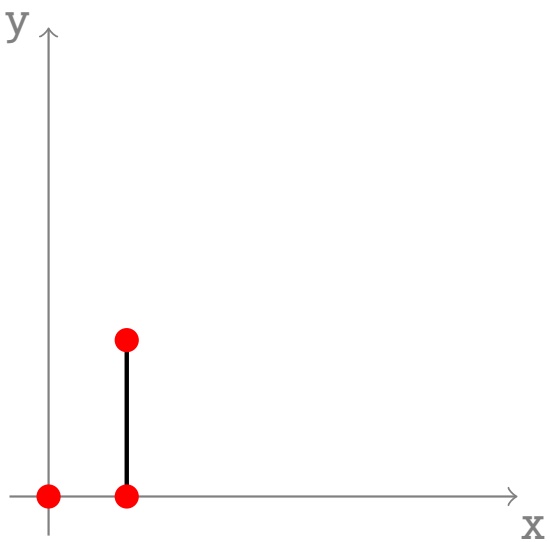


```
x = 0; y = 0;
```
$\{x = 0 \wedge y = 0\} \uplus \{x = 1 \wedge 0 \leqslant y \leqslant 2\}$

```
while (x<6) {
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```
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$\{x = 0 \wedge y = 2\}$
```
  };
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```
  x = x+1;
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```
}
```

Polyhedral analysis

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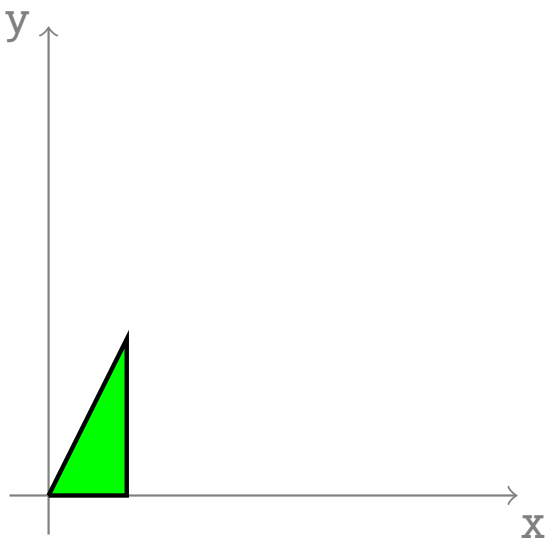


```
x = 0; y = 0;
```
$\{x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x\}$

```
while (x<6) {
  if (?) {
```
$\{x = 0 \wedge y = 0\}$

```
    y = y+2;
```
$\{x = 0 \wedge y = 2\}$

```
  };
```
$\{x = 0 \wedge 0 \leqslant y \leqslant 2\}$

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  x = x+1;
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```
}
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Polyhedral analysis

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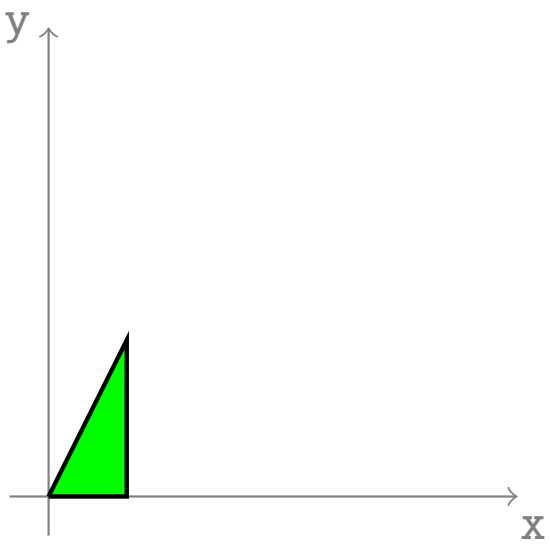


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$\{x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x\}$

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    y = y+2;
```
$\{x = 0 \wedge y = 2\}$

```
  };
```
$\{x = 0 \wedge 0 \leqslant y \leqslant 2\}$

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}
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Polyhedral analysis

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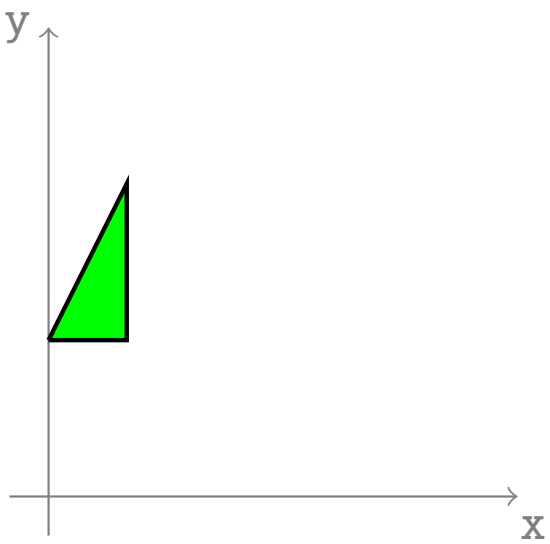


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Polyhedral analysis

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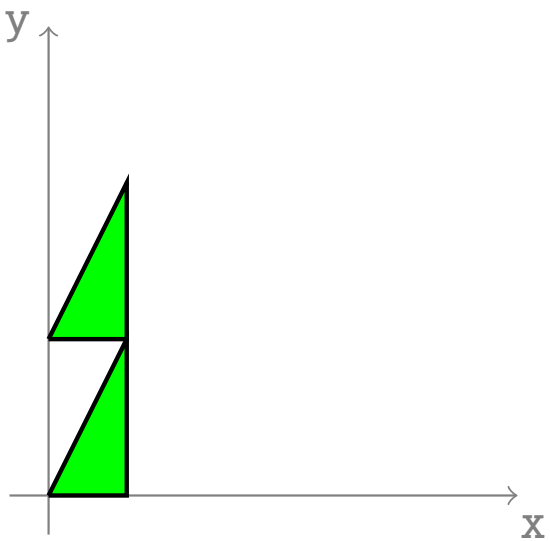


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```
}
```

Polyhedral analysis

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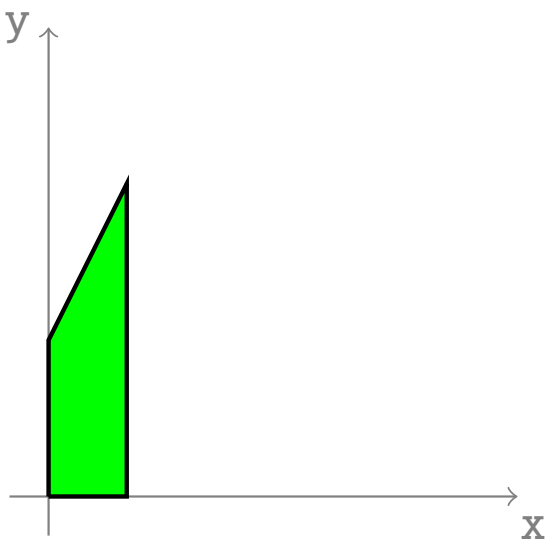


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Polyhedral analysis

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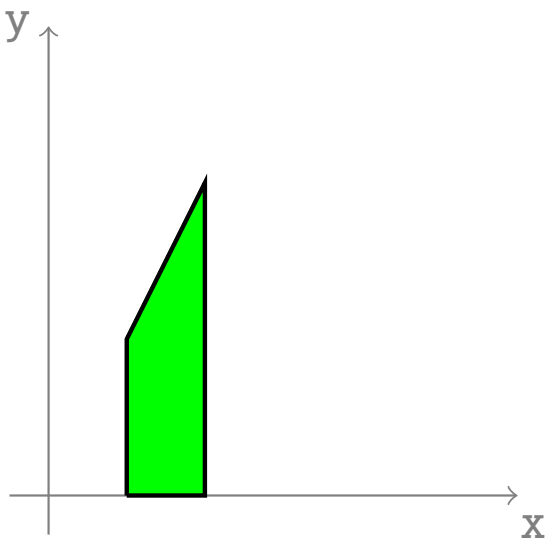


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```
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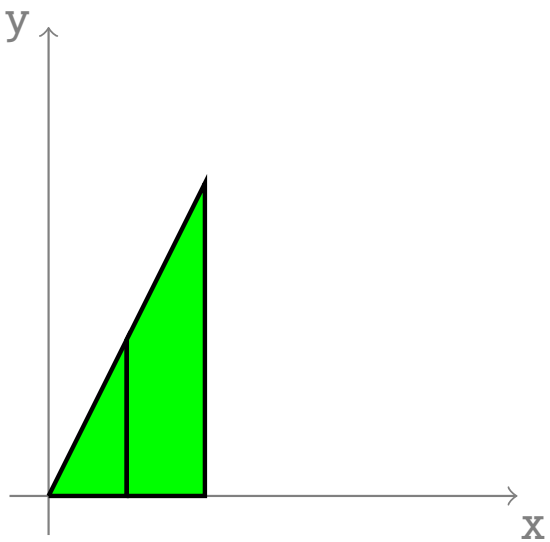
$\{1 \leqslant x \leqslant 2 \wedge 0 \leqslant y \leqslant 2x\}$

```
}
```

Polyhedral analysis

State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.



At loop headers, we use heuristics (widening) to ensure finite convergence.

```
x = 0; y = 0;
```
$\{x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x\}$
$\nabla \{x \leqslant 2 \wedge 0 \leqslant y \leqslant 2x\}$
```
while (x<6) {
  if (?) {
```
$\{x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x\}$
```
    y = y+2;
```
$\{x \leqslant 1 \wedge 2 \leqslant y \leqslant 2x+2\}$
```
  };
```
$\{0 \leqslant x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x+2\}$
```
  x = x+1;
```
$\{1 \leqslant x \leqslant 2 \wedge 0 \leqslant y \leqslant 2x\}$
```
}
```

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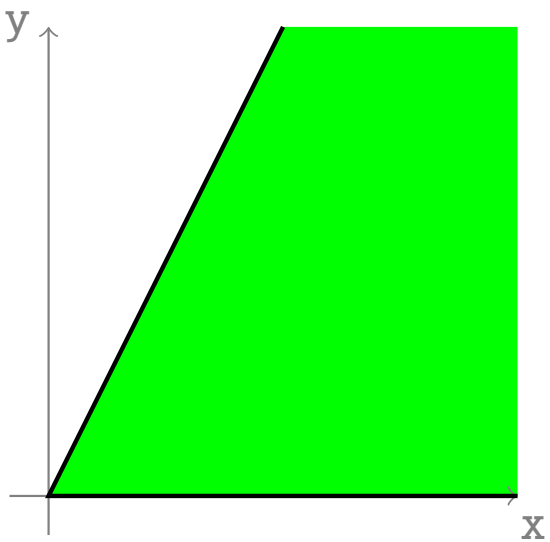


At loop headers, we use heuristics (widening) to ensure finite convergence.

```
x = 0; y = 0;
```
$\{0 \leqslant y \leqslant 2x\}$

```
while (x<6) {
  if (?) {
```
$\{x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x\}$

```
    y = y+2;
```
$\{x \leqslant 1 \wedge 2 \leqslant y \leqslant 2x+2\}$

```
  };
```
$\{0 \leqslant x \leqslant 1 \wedge 0 \leqslant y \leqslant 2x+2\}$

```
  x = x+1;
```
$\{1 \leqslant x \leqslant 2 \wedge 0 \leqslant y \leqslant 2x\}$

```
}
```

Polyhedral analysis

State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.

By propagation we obtain a post-fixpoint

```
x = 0; y = 0;
```
$\{0 \leqslant y \leqslant 2x\}$

```
while (x<6) {
  if (?) {
```
$\{0 \leqslant y \leqslant 2x \wedge x \leqslant 5\}$

```
    y = y+2;
```
$\{2 \leqslant y \leqslant 2x+2 \wedge x \leqslant 5\}$

```
  };
```
$\{0 \leqslant y \leqslant 2x+2 \wedge 0 \leqslant x \leqslant 5\}$

```
  x = x+1;
```
$\{0 \leqslant y \leqslant 2x \wedge 1 \leqslant x \leqslant 6\}$

```
}
```
$\{0 \leqslant y \leqslant 2x \wedge 6 \leqslant x\}$

Polyhedral analysis

State properties are over-approximated by convex polyhedra in $\mathbb{Q}^2$.

By propagation we obtain a post-fixpoint which is enhanced by downward iteration.

```
x = 0; y = 0;
```
$\{0 \leqslant y \leqslant 2x \wedge x \leqslant 6\}$

```
while (x<6) {
  if (?) {
```
$\{0 \leqslant y \leqslant 2x \wedge x \leqslant 5\}$

```
    y = y+2;
```
$\{2 \leqslant y \leqslant 2x+2 \wedge x \leqslant 5\}$

```
  };
```
$\{0 \leqslant y \leqslant 2x+2 \wedge 0 \leqslant x \leqslant 5\}$

```
  x = x+1;
```
$\{0 \leqslant y \leqslant 2x \wedge 1 \leqslant x \leqslant 6\}$

```
}
```
$\{0 \leqslant y \leqslant 2x \wedge 6 = x\}$

Polyhedral analysis

A more complex example.

The analysis accepts to replace some constants by parameters.

```
x = 0; y = A;
```
$\{A \leqslant y \leqslant 2x + A \wedge x \leqslant N\}$

```
while (x<N) {
  if (?) {
```
$\{A \leqslant y \leqslant 2x + A \ \wedge \ x \leqslant N - 1\}$

```
    y = y+2;
```
$\{A + 2 \leqslant y \leqslant 2x + A + 2 \ \wedge \ x \leqslant N - 1\}$

```
  };
```
$\{A \leqslant y \leqslant 2x + A + 2 \ \wedge \ 0 \leqslant x \leqslant N - 1\}$

```
  x = x+1;
```
$\{A \leqslant y \leqslant 2x + A \ \wedge \ 1 \leqslant x \leqslant N\}$

```
}
```
$\{A \leqslant y \leqslant 2x + A \ \wedge \ N = x\}$

The four polyhedra operations

- $\uplus \in \mathbb{P}_n \times \mathbb{P}_n \to \mathbb{P}_n$: convex union
  - over-approximates the concrete union at junction points
- $\cap \in \mathbb{P}_n \times \mathbb{P}_n \to \mathbb{P}_n$: intersection
  - over-approximates the concrete intersection after a conditional instruction
- $[\![\mathtt{x} := e]\!] \in \mathbb{P}_n \to \mathbb{P}_n$: affine transformation
  - over-approximates the assignment of a variable by a linear expression
- $\triangledown \in \mathbb{P}_n \times \mathbb{P}_n \to \mathbb{P}_n$: widening
  - ensures (and accelerates) convergence of (post-)fixpoint iteration
  - includes heuristics to infer loop invariants

```
x = 0; y = 0;
```
$P_0 = [\![\mathtt{y} := \mathtt{0}]\!]\,[\![\mathtt{x} := \mathtt{0}]\!]\,(\mathbb{Q}^2) \triangledown P_4$

```
while (x<6) {
  if (?) {
```
$P_1 = P_0 \cap \{x < 6\}$

```
    y = y+2;
```
$P_2 = [\![\mathtt{y} := \mathtt{y} + \mathtt{2}]\!]\,(P_1)$

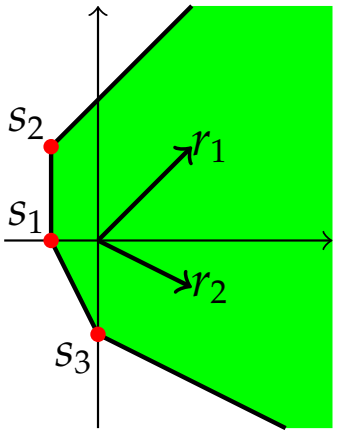
```
  };
```
$P_3 = P_1 \uplus P_2$

```
  x = x+1;
```
$P_4 = [\![\mathtt{x} := \mathtt{x} + \mathtt{1}]\!]\,(P_3)$

```
}
```
$P_5 = P_0 \cap \{x \geqslant 6\}$

Library for manipulating polyhedra

- Parma Polyhedra Library [1] (PPL), NewPolka : complex C/C++ libraries
- They rely on the Double Description Method
  - polyhedra are managed using two representations in parallel



  - by set of inequalities

$$P = \left\{ (x,y) \in \mathbb{Q}^2 \;\middle|\; \begin{array}{l} x \geqslant -1 \\ x - y \geqslant -3 \\ 2x + y \geqslant -2 \\ x + 2y \geqslant -4 \end{array} \right\}$$

  - by set of generators

$$P = \left\{ \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 + \mu_1 r_1 + \mu_2 r_2 \in \mathbb{Q}^2 \;\middle|\; \begin{array}{l} \lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2 \in \mathbb{R}^+ \\ \lambda_1 + \lambda_2 + \lambda_3 = 1 \end{array} \right\}$$

  - operations efficiency strongly depends on the chosen representations, so they keep both

1. Previous tutorial on polyhedra partially comes from `http://www.cs.unipr.it/ppl/`

Outline

1. Introduction
2. Control flow graph
3. Collecting semantics
4. Approximate analysis : an informal presentation
5. Abstraction by intervals
6. Widening and narrowing
7. Galois connections
8. Polyhedral abstract interpretation
9. References

References (1)

A few articles

- a short formal introduction :

  P. Cousot and R. Cousot. Basic Concepts of Abstract Interpretation. `http://www.di.ens.fr/~cousot/COUSOTpapers/WCC04.shtml`

- technical but very complete (the logic programming part is optional) :

  P. Cousot and R. Cousot. Abstract Interpretation and Application to Logic Programs. `http://www.di.ens.fr/~cousot/COUSOTpapers/JLP92.shtml`

- application of abstract interpretation to verify Airbus flight commands :

  P. Cousot, R. Cousot, J. Feret, L. Mauborgne, A. Miné, D. Monniaux, and X. Rival. The ASTRÉE Analyser. `http://www.di.ens.fr/~cousot/COUSOTpapers/ESOP05.shtml`

References (2)

On the web :

- informal presentation of AI with nice pictures
  http://www.di.ens.fr/~cousot/AI/IntroAbsInt.html
- a short abstract of various works around AI
  http://www.di.ens.fr/~cousot/AI/
- very complete lecture notes
  http://web.mit.edu/afs/athena.mit.edu/course/16/16.399/www/